CONLEY COMPLEXES AND CONNECTION MATRICES IN COMBINATORIAL TOPOLOGICAL DYNAMICS

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1. Introduction

Connection matrices have been introduced by R. Franzosa [7] as an algebraic topological tool in the study of Morse decompositions of flows on locally compact metric spaces. As observed by Robbin and Salamon [11], in the setting of field coefficients the algebraic part of the construction of connection matrix may be decoupled from the dynamical part by defining connection matrices for lattice filtered chain complexes and applying this general concept to the lattice of attracting neighboorhoods. Harker, Mischaikow and Spendlove [8] expand these ideas by introducing what they call a Conley complex of a poset-graded chain complex or lattice-filtered chain complex. This is a poset graded chain complex chain homotopic to the given one whose boundary map vanishes on the diagonal. They prove that Conley complex is unique up to a chain graded isomorphism. They define the connection matrix of a poset-graded chain complex or lattice-filtered chain complex as the boundary operator of a Conley complex. Since chain isomorphic complexes may differ in their boundary operators, the connection matrix need not be unique despite the fact that Conley complex is unique up to isomorphism.

In this note we apply the ideas of [11, 8] to define connection matrices for Morse decompositions of combinatorial multivector fields [10], an extension of Forman's combinatorial vector fields [5, 6]. Combinatorial multivector fields may be constructed from clouds of vectors [10, 2]; hence, they constitute a natural tool to analyze and classify dynamical data. The importance of connection matrices in this context, similarly to the case of flows, lies in the fact that a non-zero entry in the connection matrix implies the existence of a heteroclinic connection between the respective Morse sets. Moreover, it is natural to expect that the Conley complex may be helpful in classifying dynamical data.

We present an example that also in the combinatorial setting connection matrices need not be unique. But, we prove that they are unique in the case of Morse decomposition of a gradient combinatorial vector field. We also indicate some relations between persistence [3], combinatorial vector fields [5] and Conley complexes [8].

2. Main result

A Lefschetz complex (see [10] for the definition), originally defined by S. Lefschetz and called a cell complex in [8], is an abstraction of a finite combinatorial complex such as simplicial complex or cubical complex. A Lefschetz complex consists of a set of cells X and a map κ which assigns to every pair of cells a ring element called incidence coefficient. The incidence coefficient encodes the face relation between cells. Cells constitute a natural basis of the associated chain complex $C(X)$ with boundary operator defined in terms of the incidence coefficents. In this note we assume that incidence coefficients are from a fixed field $\mathbb{F}$.

A remarkable feature of every Lefschetz complex is that the face relation in X induces a T_0 Alexandrov topology $\mathcal{T}_X$ on X. This makes every Lefschetz complex X a finite topological space $(X, \mathcal{T}_X)$.

A combinatorial multivector field $\mathcal{V}$ on a Lefschetz complex X, originally defined in [10] and in this note considered in a weaker version introduced in [2] (see also [9]), is a partition of X into non-empty, locally closed sets (see [4, Sec. 2.7.1, pg 112]) in the topology $\mathcal{T}_X$. The elements of the partition are called *multivectors*. A multivector is called a *vector* if it has no more than two elements.

Research of M.M. was partially supported by the Polish National Science Center under Maestro Grant No. 2014/14/A/ST1/00453. T.W. was partially supported by NSF grants DMS-1114923 and DMS-1407087.

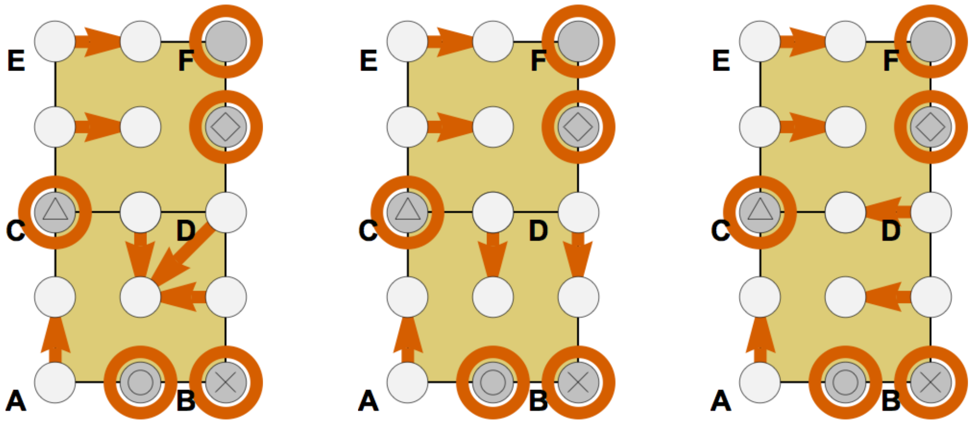

FIGURE 1. A multivector field (left) and its two combinatorial vector fields (middle and right).

In this case it has the form $V = \{V^-, V^+\}$ where either $V^- = V^+$ or V^- is a face of V^+ of codimension one.

A combinatorial multivector field $\mathcal{V}$ on a Lefschetz complex X induces a dynamical system on X. This, in particular, means that one can define isolated invariant sets, attractors, repellers and Morse decompostions [10]. For each Morse decomposition $\mathcal{M}$ there is a lattice of attracting neighbourhoods which induces a lattice filtered chain complex. In particular, one can associate with $\mathcal{M}$ the Conley complex and a non-empty collection of connection matrices. As we show in the next section, the connection matrix need not be unique. But, we prove the following theorem.

Theorem 2.1. Assume $\mathcal{V}$ is a gradient combinatorial vector field on a Lefschetz complex X. Then, the Morse decomposition consisting of all the critical cells of $\mathcal{V}$ has precisely one connection matrix. It coincides with the matrix of the boundary operator of the associated Morse complex.

3. AN EXAMPLE

Three examples of a combinatorial multivector field are presented in in Figure 1. The middle and right example are actually combinatorial vector fields, since there are no multivectors of cardinality greater than two. All three examples have the same collection of critical cells $\mathcal{M} := \{B, C, F, AB, DF\}$ and $\mathcal{M}$ is a Morse decomposition for all of them. One can verify that the left example has two connection matrices with coefficients in $\mathbb{Z}_2$:

$$C_1 := \begin{array}{c|ccccc} & B & C & F & AB & DF \\ \hline B & 0 & 0 & 0 & 1 & 1 \\ C & 0 & 0 & 0 & 1 & 0 \\ F & 0 & 0 & 0 & 0 & 1 \\ AB & 0 & 0 & 0 & 0 & 0 \\ DF & 0 & 0 & 0 & 0 & 0 \end{array} \quad \text{and} \quad C_2 := \begin{array}{c|ccccc} & B & C & F & AB & DF \\ \hline B & 0 & 0 & 0 & 1 & 0 \\ C & 0 & 0 & 0 & 1 & 1 \\ F & 0 & 0 & 0 & 0 & 1 \\ AB & 0 & 0 & 0 & 0 & 0 \\ DF & 0 & 0 & 0 & 0 & 0 \end{array} .$$

Hence, as in the case of classical dynamical systems, connection matrices in the combinatorial setting need not be unique. However, as Theorem 2.1 implies, matrix C_1 is the unique matrix of the combinatorial multivector field in the middle and matrix C_2 is the unique matrix of the combinatorial multivector field in the right of Figure 1. Note that there are examples that the connection matrix need not be unique also for non-gradient combinatorial vector fields.

4. RELATION TO PERSISTENCE.

It is known that homological persistence [3] may be phrased in terms of combinatorial Morse theory [1]. This observation may be extended to Conley complexes as follows. Assume that $\mathcal{X} = \{X_0, X_1, \ldots X_n\}$ is a filtration of a Lefschetz complex X, that is $\emptyset = X_0 \subset X_1 \subset \cdots \subset X_n = X$ is a tower of $\mathcal{T}_X$-closed subcomplexes of X. For each $x \in X$ let $t(x) := \min\{\, i \mid x \in X_i \,\}$ denote the time of appearance of x in the filtration $\mathcal{X}$. Denote by $D(\mathcal{X})$ the persistence diagram of the associated filtration of chain complexes $0 = C(X_0) \subset C(X_1) \subset \cdots \subset C(X_n)$. Recall that the persistence diagram is a multiset consisting of pairs (p, q) where p is the birth time of a homology class and q is its death time or infinity if the class never dies.

We say that a combinatorial vector field $\mathcal{V}$ on X is a *persistence combinatorial vector field* with respect to the filtration $\mathcal{X}$ if the map $\alpha : \mathcal{V} \to D(\mathcal{X})$ given by

$$\alpha(V) := \begin{cases} (t(V^-), t(V^+)) & \text{if } V^- \neq V^+, \\ (t(V^-), \infty) & \text{if } V^- = V^+ \end{cases}$$

is a bijection of multisets.

The filtration $\mathcal{X}$ is obviously a lattice with respect to union and intersection. This makes $C(X)$ a filtered chain complex and allows one to associate with $\mathcal{X}$ a Conley complex $\operatorname{Con}(\mathcal{X})$.

Theorem 4.1. Given a filtration $\mathcal{X}$ of a Lefschetz complex X there is another Lefschetz complex $\bar{X}$ and a bijection $\theta : X \ni x \mapsto \bar{x} \in \bar{X}$ such that

(i) $\bar{\mathcal{X}} := \{\theta(X_0), \theta(X_1), \ldots, \theta(X_n)\}$ is a filtration of $\bar{X}$,
(ii) θ induces a chain isomorphism of filtered chain complexes $C(X)$ and $C(\bar{X})$,
(iii) $\bar{X}$ admits a persistence combinatorial vector field with respect to $\bar{\mathcal{X}}$,
(iv) the Conley complexes of $\mathcal{X}$ and $\bar{\mathcal{X}}$ coincide,
(v) in particular, persistence diagrams of $\mathcal{X}$ and $\bar{\mathcal{X}}$ coincide.

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