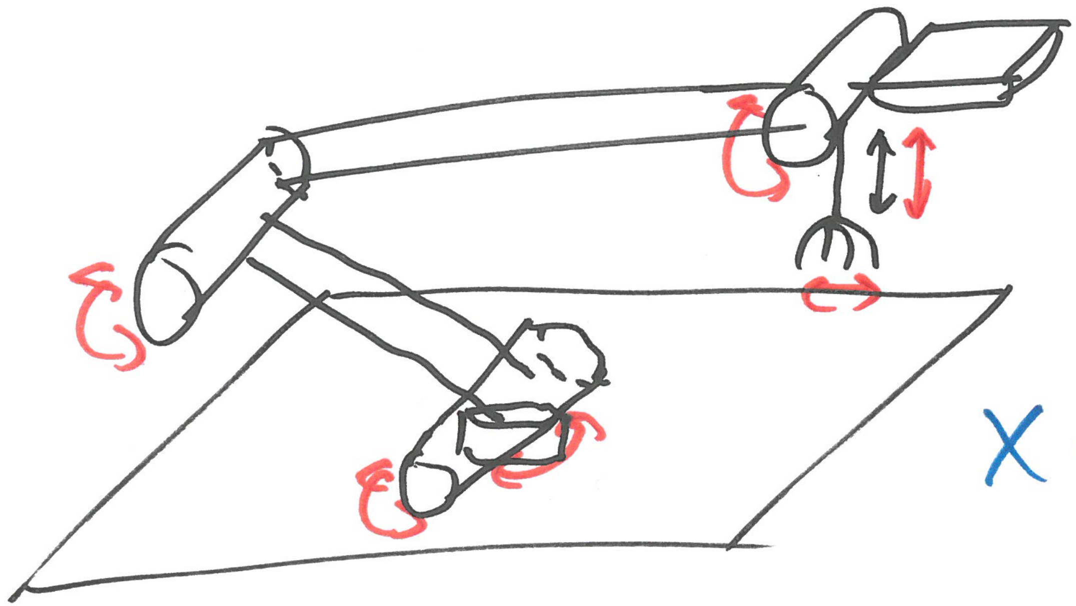
Topological aspects of robot motion planning I

Mars Insight IDA



6 DOF

Configuration space

$X \subseteq S^1 \times S^1 \times S^1 \times S^1 \times [a,b] \times [c,d]$

Motion planning problem Given initial and target configurations $A, B \in X$, output a path in X from A to B.

<u>Path fibration</u> $PX = \{\gamma : I := [0,1] \to X \text{ cts}\}$ (compact-open)

$$\pi : PX \longrightarrow X \times X$$
$$\gamma \longmapsto (\gamma(0), \gamma(1))$$

<u>Motion planner</u> $s : X \times X \longrightarrow PX$ section ($\pi \circ s = \mathrm{Id}_{X \times X}$)

(not necc. continuous)

<u>Thm (Farber)</u> $\exists$ cts motion planner in X

$\Longleftrightarrow$ X contractible.

<u>Defn</u> (Farber) <u>Topological complexity</u> of X is

$$TC(X) = \min\{k \mid X \times X = U_1 \cup \dots \cup U_k,\ U_i \underset{\text{open}}{\subseteq} X \times X,$$
$$\exists\, s_i : U_i \to PX,\ \pi \circ s_i = \mathrm{incl} : U_i \hookrightarrow X \times X\}$$

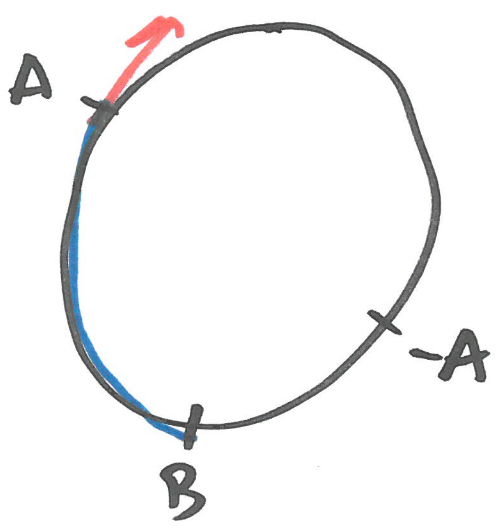
Remark If X is ENR, can replace open cover by partition into ENRs.

Exercise If $X \simeq Y$ then $TC(X) = TC(Y)$

Example spheres S^n

Can always take $U_1 = \{(A,B) \mid A \neq -B\} \subseteq S^n \times S^n$

$s_1(A,B)$ = shortest geodesic



For n odd : Take a nowhere zero v.f. v on S^n.

On $U_2 = \{(A,-A)\}$,

$s_2(A,-A)$ = geodesic arc in dir$^{\underline{n}}$ $v(A)$

For <u>n even</u> : Take a v.f. v with one zero $A_0 \in S^n$

$U_2 = \{(A, -A) \mid A \neq A_0\}$, s_2 as above

$U_3 = \{(A_0, -A_0)\}$, s_3 any path.

S^n not contractible $\Rightarrow TC(S^n) > 1$.

Hence $TC(S^n) = \begin{cases} 2, & n \text{ odd} \\ 2 \text{ or } 3, & n \text{ even}. \end{cases}$

<u>Cohomology</u> gives lower bounds.

<u>Def^n</u> A <u>zero-divisor</u> is an element ~~of~~

$$z \in \ker(\Delta^* : H^*(X \times X) \to H^*(X))$$

where $\Delta : X \to X \times X$ diagonal map
$\qquad x \mapsto (x, x)$

Thm (Farber) If $\exists$ zero-divisors $z_1, \dots, z_k$ such that $z_1 \cup \dots \cup z_k \neq 0 \in H^*(X \times X)$, then $TC(X) > k$.

Proof Suppose $TC(X) \leq k$, $X \times X = U_1 \cup \dots \cup U_k$

$$\begin{tikzcd} & PX \arrow[d, "\pi"] & X \arrow[l, "\simeq"'] \arrow[dl, "\Delta"] \\ U_i \arrow[ur, "s_i"] \arrow[r, hook] & X \times X & \end{tikzcd}$$

$$\begin{array}{ccccc} H^*(X \times X, U_i) & \longrightarrow & H^*(X \times X) & \longrightarrow & H^*(U_i) \\ \tilde{z}_i & \longmapsto & z_i & \longmapsto & 0 \end{array}$$

$$z_1 \cup \dots \cup z_k = j^*(\underbrace{\tilde{z}_1 \cup \dots \cup \tilde{z}_k}_{\in\, H^*(X \times X,\, U_1 \cup \dots \cup U_k) = 0}) = 0$$

$\square$

__Ex__ (spheres-ctd) $H^*(S^n) = \mathbb{Z}[a]/(a^2)$, $|a| = n$

In $H^*(S^n \times S^n) \cong H^*(S^n) \otimes H^*(S^n)$

$$\begin{aligned} (1\times a - a\times 1)^2 &= -a\times a - (-1)^n a \times a \\ &= \begin{cases} 0, & n \text{ odd} \\ \underset{\neq 0}{-2a\times a}, & n \text{ even} \end{cases} \end{aligned}$$

$$1\times a - a \times 1 \;\xrightarrow{\Delta^*}\; 1\cup a - a \cup 1$$

Thus $TC(S^{\text{even}}) = 3$ $\square$

__Obstruction theory__ gives the general upper bound

$$TC(X) < \frac{2 \dim X + 1}{s+1} + 1$$

if X is an s-connected CW complex.

<u>Ex</u> $TC(\mathbb{C}P^n) < \frac{2(2n)+1}{1+1} + 1 = 2n + \frac{1}{2} + 1$

$H^*(\mathbb{C}P^n) = \mathbb{Z}[x]/(x^{n+1})$, $|x| = 2$

$(1 \times x - x \times 1)^{2n} = (-1)^n \binom{2n}{n} x^n \times x^n \neq 0$

so $\underline{TC(\mathbb{C}P^n) = 2n+1}$ $\square$

Also have $cat(X) \leq TC(X) \leq cat(X \times X)$

where $cat(X) = \min\{k \mid X = U_1 \cup \dots \cup U_k,\ U_i \subseteq X$ open, $U_i \hookrightarrow X$ null-homotopic$\}$.

— Lusternik-Schnirelmann category.

Problems & Variations

Problem (Farber) π (discrete) group

$$TC(\pi) := TC(K(\pi, 1))$$

Describe $TC(\pi)$ in terms of algebraic invariants of π.

Eilenberg–Ganea, Stallings, Swan: $\mathrm{cat}(\pi) =$ cohomological dimension of π

Monoidal TC (Iwase–Sakai)

Considers covers by $U_i \supseteq \Delta(X)$ with $s_i(A, A) =$ constant at A.

$$TC(X) \leq TC^M(X)$$

Problem Does $TC^M(X) = TC(X)$?
(for loc. fte simplicial complexes)

Directed TC (Goubault)

Symmetrized TC (Basabe-González-Rudyak-Tanaki)

$$TC(X) \neq \mathrm{cat}\left(X \times X / \Delta X\right)$$

I <u>Directed TC</u> (j.w. Ayşe Borat)

<u>Def$^{\underline{n}}$</u> (M. Grandis) A <u>directed space</u> or <u>d-space</u> $(X, \overrightarrow{PX})$ is a space X and a collection of <u>dipaths</u> $\overrightarrow{PX} \subseteq PX$ satisfying:

1. constants $\subseteq \overrightarrow{PX}$;
2. $\overrightarrow{PX}$ closed under concat$^{\underline{n}}$;
3. " " " precomp$^{\underline{n}}$ with non-decreasing maps $I \to I$

<u>E.g</u> · $(X, \leq)$ pospace, $\overrightarrow{PX} = \{ \gamma : (I, \leq) \to (X, \leq)$ cts, order-preserving $\}$

· $\dot{x}(t) = F(x(t), u(t), t)$ with controls $u(t) \subseteq U \subseteq \mathbb{R}^p$
Trajectories generate d-space

$$\begin{tikzcd} \overrightarrow{PX} \arrow[d, "\chi"'] & \subseteq & PX \arrow[d, "\pi"] \\ \pi(\overrightarrow{PX}) =: \vec{\Gamma}_X & \subseteq & X \times X \end{tikzcd}$$

<u>Defn</u> (E. Goubault '17) Directed TC of $(X, \overrightarrow{PX})$ is

$$\overrightarrow{TC}(X) = \min\{k \mid \vec{\Gamma}_X = A_1 \sqcup \dots \sqcup A_k,\ A_i \subseteq \vec{\Gamma}_X \text{ ENR} \\ \exists\, \sigma_i : A_i \to \overrightarrow{PX} \text{ s.t. } \chi \circ \sigma_i = \mathrm{incl}\}$$

<u>Remarks</u>

- χ not a fibration
- $\overrightarrow{TC}(X)$ directed homotopy invariant.
- Lower bounds? Directed cohomology?

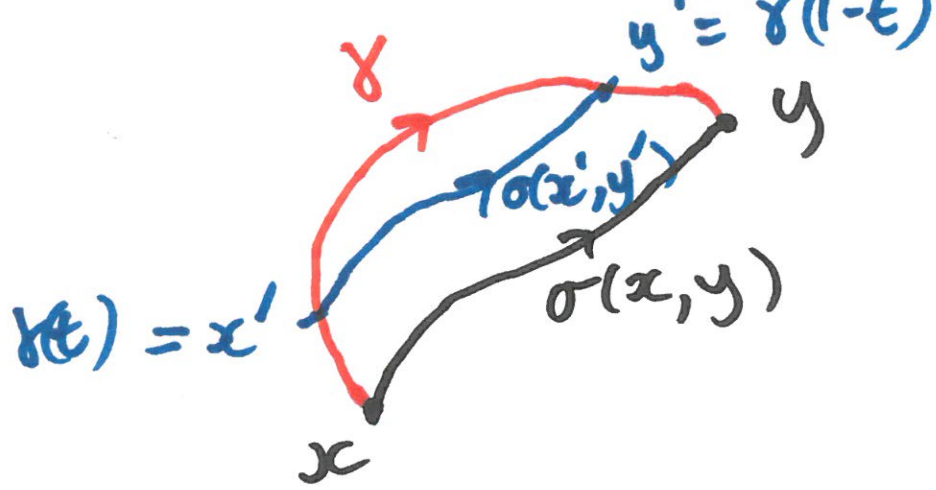
Lemma (Garbault) If $\overrightarrow{TC}(\mathcal{X}) = 1$, then each fibre $\overrightarrow{PX}(x,y)$ of $\mathcal{X}$ is contractible.

'**Proof**' Let $\sigma : \Gamma_{\mathcal{X}} \to \overrightarrow{PX}$ be a section of $\mathcal{X}$.

Given $(x,y) \in \Gamma_{\mathcal{X}}$, want

$$\{\sigma(x,y)\} \underset{\text{const}}{\overset{\hookrightarrow}{\leftarrow}} \overrightarrow{PX}(x,y)$$

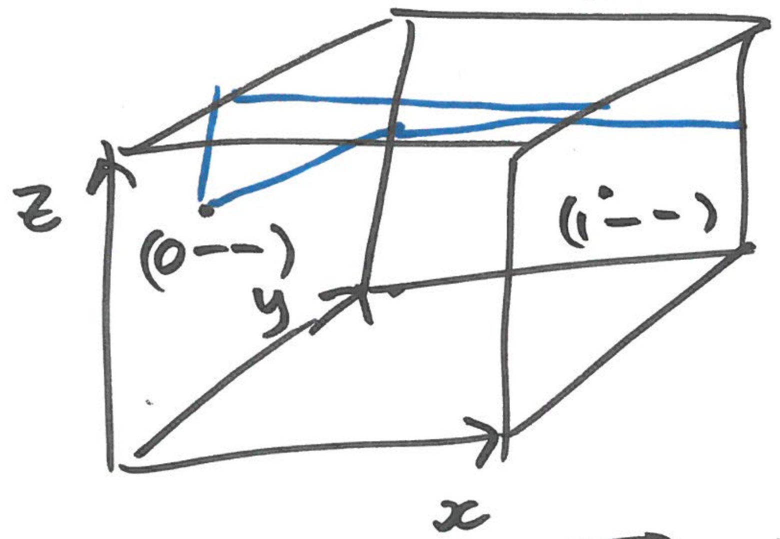
to be h. inverses



$\square$

Defn Directed sphere $\vec{S}^n$ is $\partial I^{n+1} \subseteq \mathbb{R}^{n+1}$ with dipaths non-decreasing in every co-ord.

$\vec{S}^2$

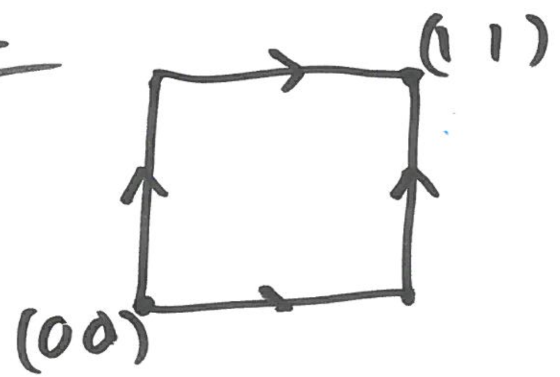


$- \in (0,1)$

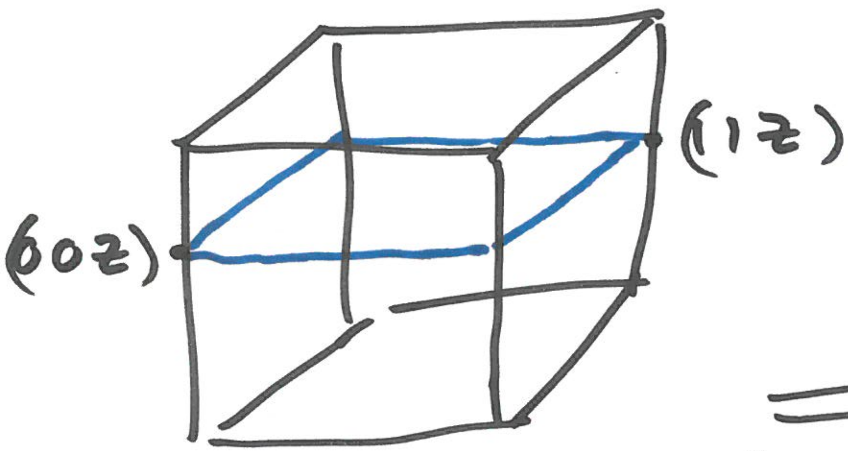
$(0--, 1--) \notin \Gamma_{\vec{S}^n}$

Thm (Borat, G. '18) $\vec{TC}(\vec{S}^n) = 2 \quad \forall n \geq 1$.

Proof



$\vec{P}S^1(00, 11) \not\simeq *$



$$\overrightarrow{PS^n}(00x_2\ldots x_n,\ 11x_2\ldots x_n)$$
$$\cong \overrightarrow{PS^1}(00,11) \not\simeq *$$
$$\underset{\text{Lemma}}{\Longrightarrow}\ \overrightarrow{TC}(S^n) > 1 \quad \forall\, n \geq 1$$

Now give partition $\Gamma_{\vec{S^n}} = A_1 \sqcup A_2$ with dir. motion planners.

<u>Idea</u> Note $\overrightarrow{TC}(I^{n+1}) = 1$: may take

$\sigma_1 =$ "Increase x_0 to y_0, then x_1 to $y_1, \ldots,$ then x_n to y_n"

<u>or</u> $\sigma_2 =$ "Increase x_n to $y_n, \ldots\ldots,$ then x_0 to y_0"

<u>Observe</u> $\sigma_1(\underline{x}, \underline{y})$ enters $\mathrm{int}(I^{n+1})$ iff

$\exists\, 0 \leq k \leq n : x_k < y_k$ & $\cancel{y_0, \ldots, y_{k-1}, x_{k+1}, \ldots, x_n}$ $\in (0,1)$
$x_0, \ldots, x_{k-1}, y_{k+1}, \ldots, y_n$

$\sigma_2(\underline{x}, \underline{y})$ enters $\mathrm{int}(I^{n+1})$ iff

$\exists\, 0 \leq j \leq n \,:\, x_j < y_j$ & $y_0, \ldots, y_{j-1}, x_{j+1}, \ldots, x_n \in (0,1)$

Show either $\sigma_1(\underline{x}, \underline{y})$ or $\sigma_2(\underline{x}, \underline{y}) \subseteq \partial I^{n+1}$

$\forall (\underline{x}, \underline{y}) \in \Gamma_{\vec{S^n}}$ $\square$

Π – Symmetrized TC

X any space

Path fibration has C_2-symmetry

$$\begin{array}{ll} PX \circlearrowleft & \gamma \mapsto \overline{\gamma} \\ \downarrow \pi & \\ X \times X \circlearrowleft & (A,B) \mapsto (B,A) \end{array}$$

__Def$^{\underline{n}}$__ (Basabe, González, Rudyak, Tamaki)

__Symmetrized TC__ of X is

$$TC^{\Sigma}(X) = \min \left\{ k \;\middle|\; \begin{array}{l} X \times X = U_1 \cup \dots \cup U_k,\ U_i \subseteq X \times X \text{ open, } C_2\text{-}\underline{\text{invariant}} \\ \exists\ s_i : U_i \to PX \text{ cts } C_2\text{-}\underline{\text{equivariant}} \\ \pi \circ s_i = \text{incl}. \end{array} \right\}$$

__Observe__ $TC(X) \leq TC^{\Sigma}(X)$

__Exercises__

- $X \simeq Y \Rightarrow TC^{\Sigma}(X) = TC^{\Sigma}(Y)$
- $TC^{\Sigma}(S^n) \leq 3 \quad \forall n$

(so $\quad = \begin{cases} 3, & n \text{ even} \\ \underset{\text{or } 3}{2}, & n \text{ odd} \end{cases}$

Lower bounds from <u>equivariant</u> <u>cohomology</u>

- Borel $H^*(EC_2 \times_{C_2} (X \times X))$ — too simple!
- Bredon $H^*_{C_2}(X \times X; M)$ — too hard!
- Symm square $H^*(SP^2(X))$ — just right!

$$SP^2(X) = X \times X / C_2$$

<u>Def</u>: Let $dX \subseteq SP^2(X)$ the diagonal. A <u>symm. zero divisor</u> is an elt $z \in \ker\left(H^*(SP^2(X)) \to H^*(dX)\right)$

<u>Thm</u> If $\exists$ symm. z.d's $z_1, \dots, z_k$: $0 \neq z_1 \cup \dots \cup z_k \in H^*(SP^2(X))$, then $TC^{\Sigma}(X) > k$

Nakaoka '56 $\Rightarrow$ for $n \geq 2$, $\exists$ symm. z.d.

$z \in H^n(SP^2(S^n))$ with $0 \neq z^2$

$\Rightarrow TC^{\Sigma}(S^n) = 3$ for $n \geq 2$.

Note $SP^2(S^1) \cong$ Möb — no products in H^*!

Define $TC^{\Sigma,M}(X) =$ "symm. monoidal TC"

Thm (G. '17) If X paracompact ENR,

$$TC^{\Sigma,M}(X) = TC^{\Sigma}(X)$$

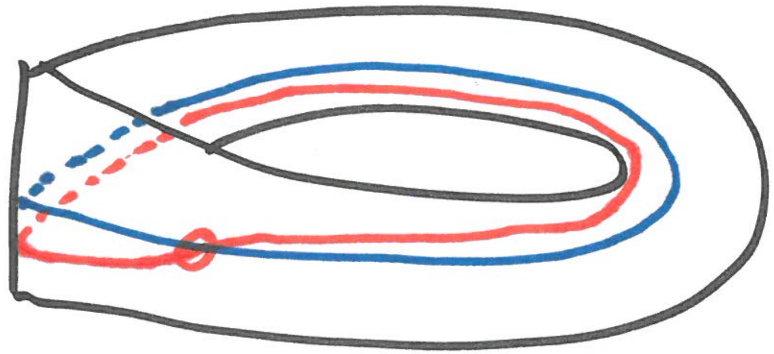
Thm If $\exists\, z_1, \dots, z_k \in H^*(SP^2(X), dX)$:

$z_1 \cup \dots \cup z_k \neq 0$, then $TC^{\Sigma,M}(X) > k$.

Corollary (G., D. Davis) $TC^{\Sigma}(S^1) = 3$

Pf In $H^*(SP^2(S^1), dX; \mathbb{Z}/2) \cong H^*(\text{Möb}, \partial\text{Möb}; \mathbb{Z}/2)$

Let $z \in H^1(\text{Möb}, \partial\text{Möb}) \cong H_1(\text{Möb})$ be PD of core circle:

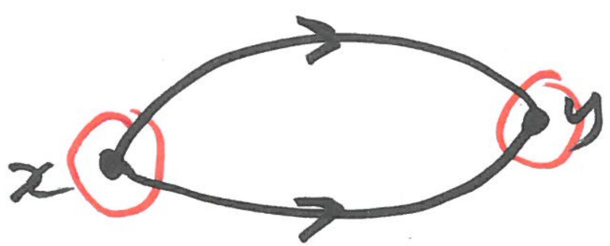


Then $z^2 \neq 0$.

$\square$

<u>Problems</u>

- compute $TC^{\Sigma}(T^2)$
- $\exists X \not\simeq *$ with $TC^{\Sigma}(X) \neq TC^{S}(X)$ symmetric TC of Farber-G.



$(x, y) \in U$