TOPOLOGICAL ASPECTS OF ROBOT MOTION PLANNING

MARK GRANT

The set of physical states of a robot or mechanical system carries the structure of a topological space, the so-called configuration space of the system. The motion planning problem of robotics then translates to the topological problem of assigning to each pair of points in configuration space a path between them. If the configuration space is not contractible, then no such assignment of a path can be found which depends continuously on the input points. From a topological perspective, a motion planning algorithm may be viewed as optimal if it minimizes the discontinuities in a suitable sense.

These observations led Michael Farber to introduce a new numerical homotopy invariant, called *topological complexity*, which quantifies the complexity of motion planning algorithms in the given configuration space [2, 3]. By now the theory of this invariant is fairly well developed, with many computations, examples and variants in the literature.

In these talks I will survey the topological complexity of motion planning algorithms, starting with basic examples and building up to recent research. If time permits I will discuss directed [1] and symmetrized [4] topological complexity of spheres, and group-theoretic lower bounds for the topological complexity of $K(\pi, 1)$ spaces [5].

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REFINING PERSISTENCE VIA ENRICHED TOPOLOGICAL SUMMARIES

JUSTIN MICHAEL CURRY

Persistent Homology can be thought of as a mapping from various spaces of data to the stratified space of persistence diagrams. By considering inverse problems for the persistence map, we are led to various enrichments of classical tools in TDA: chiral, and decorated, merge trees; Reeb graphs equipped with a (nesting) poset structure on their fibers, and braided Reeb graphs. Some of these enriched topological summaries serve a two-fold purpose: 1) they provide discrete combinatorial structures that are in bijection with certain equivalence class structures put on the fiber of the persistence map, and 2) they are themselves a computationally efficient augmentation of persistence with sharper distinguishing power than persistence alone.

As an illustration of this first purpose I will show an extension of the formula in [Curry 2017] for counting "height equivalence" classes of functions on the two-sphere. The class of functions considered here (the "space of data types" mentioned above) are restricted to height functions on the two-sphere gotten by embedding the two-sphere in $\mathbb{R}^3$ and projecting onto a fixed direction vector $v \in S^{d-1}$. Height equivalence of functions is given by considering embedded spheres up to level-set preserving isotopy. Such an equivalence relation provides a discretization of the fiber of the persistent homology *transform* (PHT) evaluated at the single direction v. In the case where the function $h_v(x) = x \cdot v$ has only N minima, $N-1$ saddles, and 1 global maximum, the formulas in [Curry 2017] generalize directly, but chirality becomes replaced with a "nesting" poset structure on the fibers of the Reeb graph. However, for embeddings that yield height functions with two or more maxima, one has to consider braided Reeb graphs that respect the nesting relationships on the fibers. If time allows it, the injectivity results of [CMT 2018] and the characterizations of non-injectivity described above will be related and further questions will be posed.

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CONFIGURATION SPACES OF HARD DISKS IN AN INFINITE STRIP

MATTHEW KAHLE

This talk is based on joint work with Robert MacPherson. We study the configuration space config(n,w) of n nonoverlapping disks of unit diameter in an infinite strip of width w. Our main result establishes the rate of growth of the Betti numbers $\beta_j[\mathrm{config}(n, w)]$ for every fixed j and w as $n \to \infty$.

We identify three regions in the (j, w) plane exhibiting qualitatively different topological behavior. We describe these regions as (1) a "gas" regime where homology is stable, (2) a "liquid" regime where homology is unstable, and (3) a "solid" regime where homology is trivial. We describe the boundaries between stable, unstable, and trivial homology for every $n \geq 3$.

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CYCLES IN RANDOM SIMPLICIAL COMPLEXES, LARGE AND SMALL

MATTHEW KAHLE

The topology of random simplicial complexes has been studied intensely for the past fifteen years or so. Some of the main topics that have been studied include: when is homology vanishing or non-vanishing, if non-vanishing how large are the Betti numbers, etc. See Chapter 22 of [1] for a recent survey.

In this talk, we are interested in a slightly more refined and geometric picture—how large are the cycles in a random complex? Of course the answer depends on how we measure the sizes of cycles and also on the model of random complex.

We will see that for the Linial–Meshulam random 2-complex, most cycles are large [2]. This inspires a proof of the existence 2-dimensional simplicial complexes with nearly optimally large homological systoles. This proof depends on the probabilistic method, and at the moment we have no idea how to construct such complexes explicitly.

On the other hand, we will also see that for a random geometric complex, all the cycles are small [3]. Here we measure the size of holes in terms of persistent homology. We show that the maximally persistent cycles are sub-logarithmic in size. This work is inspired by questions in topological data analysis, trying to separate topological signal from noise.

We will define these models of random complex as we go, and the talk will aim to be self contained.

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CLASSIFICATION OF 2D HAMILTONIAN VECTOR FIELDS AND TOPOLOGICAL FLOW DATA ANALYSIS: THEORY, COMPUTATION AND APPLICATIONS

TAKASHI SAKAJO

Fluid dynamics has been one of the important subjects of science and technologies. Numerical simulations of fluid equations play a significant role in the developments of modern infrastructure such as cars, high-speed trains, airplanes and wind turbine generators. Owing to the recent improvements of observation and measurement technologies, it is also utilized to extract useful information from ultrasonic images of cardiovascular flows and satellite images of ocean and coastal flows. On the other hand, although a large amount of visualized flow data is available, it is sometimes very difficult to express those flow patterns in words. The lack of common language among researches in multiple disciplines gives rise to an obstacle to proceed the interdisciplinary research. Moreover, with the explosion of data size obtained by such numerical simulations, observations and measurements, it is strongly desired to develop an efficient way describing flow properties and making their predictions from those massive data. To respond to these demands, we have developed a new classification theory for global streamline patterns of two-dimensional incompressible flows by making use of topology, discrete mathematics and the theory of dynamical systems.

What we have developed is a combinatorial classification for structurally stable Hamiltonian vector fields on multiply connected planar domains in the presence of a uniform flow, which is a model of two-dimensional incompressible fluid flows. The theory allows us to assign a unique sequence of letters, called *maximal words* and *regular expressions*, to every global topological structure created by the Hamiltonian vector fields [1, 4]. See the schematic of Figure 1 explaining the basic idea how the sequence of letters is assigned to streamline topologies. The conversion to maximal words and regular expressions is easy to implement, and the sequence of letters are intuitively interpretable to those who are not familiar with mathematics. An automatic conversion algorithm has already been implemented on computers as a software, and it is thus applicable to massive flow pattern data obtained by numerical simulations and/or physical measurements in fluid science, engineering and medical studies. By extracting global topological information from flow data, one is expected to figure out latent knowledge that are not recognized by experts in those fields so far. For instance, as demonstrated in [3], a certain flow functionality such as the maximum/minimum drag-to-lift ratios acting on a wing in the presence of a uniform flow is encoded as a specific sequence of letters contained commonly in maximal words and regular expressions of data-sets, which means that the sequence of letters works as a "DNA" for flows. See the article [5] for the underlying concepts in our theory. In addition, we have also developed a mathematical theory describing all possible global transitions of streamline topologies, without exceptions, through marginal structurally unstable Hamiltonian vector fields in terms of the changes of the sequence of letters [2]. Hence, by simply comparing them, we predict the change of global flow patterns that could possibly happen in future.

We will also introduce a new way of topological data analysis, called *topological flow data analysis (TFDA)*, based on the classification theory. Owing to TFDA, long-time evolutions of flows (or Hamiltonian vector fields), whose data size often exceeds more than giga-bytes, is drastically compressed into a small size of text data expressing the change of streamline topologies, which is amenable to statistical and/or time-series analysis, and machine learning for global topological information with ease. We show some applications to medical images of cardiovascular flows and flow patterns in meteorology. We also show another example illustrating that TFDA is available to create a data-driven model predicting a complex flow phenomenon.

This work is partially supported by JSPS Kakenhi(B) #18H01136 and JST Mirai. Examples of data used in the topological flow data analysis is provided by Prof. M. Nasser, Prof. M. Inatsu, Dr. K. Itatani and Dr. T. Matsumoto.

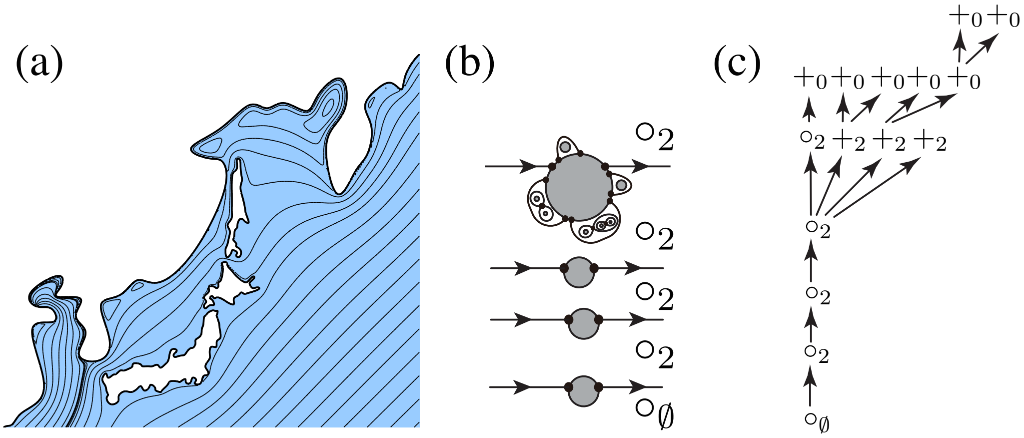

Figure 1. Encoding the maximal word and the regular expression to a numerically constructed streamline pattern around East-Asia ocean current shown in (a). Extracting topological pattern structures from the streamlines and assigning specific symbols to each domains separated by those topological streamlines as in (b), we then construct a tree structure to the adjacent relations between those domains as we see in (c). The tree structure is expressed as the unique regular expression $\circ_\emptyset(\circ_2(\circ_2(\circ_2(\circ_2(-_2), +_2(+_0, +_0), +_2(+_0, +_0(+_0, +_0)), +_2)))$. The maximal word for this streamline topology is given by $IA_2A_2A_2CCCCB_0B_0B_0$ according to [1, 4]. This process is now automatically executable with our computer software.

In the first part of my talk, starting with a brief review of the potential flow theory for those who are unfamiliar with fluid dynamics, we give the mathematics of the classification theory for structurally stable Hamiltonian vector fields. In the second part of my talk, we will explain how to implement the conversion algorithm as a computer software, and will introduce some applications of the theory to some flow problems and discuss future extensions of the theory. The contents of this talk is based on the joint works with Dr. T. Yokoyama (Kyoto University of Education), Dr. N. Nakano (Kyoto University) and Dr. T. Uda (Tohoku University).

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LEARNING GEOMETRY USING TOPOLOGY AND PERSISTENCE LANDSCAPES

PETER BUBENIK

I this talk I will give an introduction to Topological Data Analysis (TDA), which summarizes the shape of data. It is sometimes said that TDA detects the underlying topology of the data; I will argue that it is better to say that TDA captures the underlying geometry of the data. I will support this thesis with two examples: one using biological images and the other using points sampled from surfaces of constant curvature.

(Peter Bubenik) University Florida

ALGEBRAIC DISTANCES FOR PERSISTENT HOMOLOGY

PETER BUBENIK

One of the main ideas in Topological Data Analysis is to convert application data into an algebraic object called a persistence module and to calculate distances between such modules. I will introduce these constructions and describe the main examples of such distances, called Wasserstein distances. The weakest of these distances, called the bottleneck distance, has previously been described algebraically (called interleaving distance). This has led to much useful theory and applications. I will give an algebraic description of all of the Wasserstein distances and discuss their generalizations.

(Peter Bubenik) UNIVERSITY FLORIDA

COMPUTING EXPLICIT HOMOLOGY CLASSES USING DISCRETE MORSE THEORY

DMITRY FEICHTNER-KOZLOV

In this talk we shall describe a combinatorial method related to Discrete Morse Theory, which allows us to calculate explicit homology cycles. These cycles will form a basis, in the case when the critical cells are in an isolated dimension. We shall illustrate the use of this technique by several examples from combinatorial topology.

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ALGORITHMIC CANONICAL STRATIFICATIONS OF SIMPLICIAL COMPLEXES

JAY SHAH

Simplicial complexes are mathematical structures of primary significance within topological data analysis, being a model for (nice) topological spaces that can be represented and manipulated on a computer. For the purposes of TDA, we seek computationally accessible invariants of simplicial complexes. In this talk, I will describe one such invariant deriving from the theory of stratified spaces, which is the coarsest stratification of a simplicial complex such that the strata are homology manifolds, and describe an efficient algorithm for calculating this "canonical" stratification. More precisely, given the poset P of simplices of a finite abstract simplicial complex K, we may algorithmically determine the map of posets $\pi : P \to [\dim(P)]$ such that for each fiber $P_{\pi=i} \subset P$, $P_{\pi=i}$ is maximal among all open subposets $U \subset \overline{P_{\pi=i}}$ in its closure such that the restriction of the local $\mathbb{Z}$-homology sheaf of $\overline{P_{\pi=i}}$ to U is locally constant. The main new idea is to iteratively constrain the stable homotopy types of the links of simplices via Poincaré duality. This is joint work with Ryo Asai.

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HOMOLOGICAL CLUSTERING AND SIMPLICIAL CONVOLUTIONAL NEURAL NETWORKS

GARD SPREEMANN

The talk has two parts. Both relate to simplicial Laplacians, which shows promise to complement traditional tools in topological data analysis.

The first part is concerned with a generalization of spectral clustering [1, 2]. In classical spectral clustering of graphs, the vertices are first embedded in Euclidean space by means of the (low-eigenvalue) eigenvectors of the graph Laplacian, then some Euclidean clustering scheme is applied, before the result is pulled back to the graph. We introduce a similar scheme for simplicial complexes that is sensitive to the homology of the complex.

In the second part of the talk, we describe how the simplicial Laplacian allows us to define simplicial convolutional networks to perform deep learning where the input and output data are cochains on a fixed underlying simplicial complex, and where the learning is sensitive to this structure. This provides a broad generalization of regular and graph-based CNNs [3].

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TOPOLOGICAL ANALYSIS OF THE CHEMICAL SPACE: UNDERSTANDING AQUEOUS SOLUBILITY.

JACEK BRODZKI

In this talk we will present an application of topological data analysis to understand the structure of the descriptor space of molecules produced from a standard chemical informatics software. We are interested in discovering indicators of when a chemical compound is soluble in water. We have used the mapper algorithm, a TDA method that creates low-dimensional representations of data, to create a network visualization of the solubility space. While descriptors with clear chemical implications are prominent features in this space, reflecting their importance to the chemical properties, the topological analysis has uncovered new and interesting chemical properties responsible for water solubility.

We have also considered a representation of the chemical space using persistent homology applied to molecular graphs, and we have discovered that links between this chemical space and the descriptor space are in agreement with chemical heuristics.

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POINT VORTEX DYNAMICS ON MINIMAL SURFACES

YUUKI SHIMIZU

In fluid dynamics on curved surfaces, the shape of the flow field is a primary factor in determining the dynamics of the fluid flow. Some mathematical models of fluid equations on surfaces are proposed in terms of differential geometry without physical experiments [1]. Our purpose is to characterize dynamical properties derived from each of these models in terms of geometric properties of the surfaces towards validating the model with a physical experiment.

Let us consider a situation where fluid dynamics on curved surfaces can be physically realized. As we know, when we hang some wires and dip them into soap solution, a soap film spanning the wires is formed [2]. The shape of the soap film is determined by the Young-Laplace equation. Assuming the pressure difference across a film is equal to zero, we can deduce that the shape of the soap film is given as a minimal surface. The motion of fluid in the soap film is governed by the Euler equations on surfaces when we assume the fluid is incompressible and inviscid. In this talk, we treat the motion of fluid in a soap film as that of incompressible and inviscid fluid on a minimal surface. When we compare physical experiments with theoretical analysis of the Euler equations on a minimal surface, it is necessary to construct a numerical scheme. To this end, we divide this task into two parts: surface registration of a minimal surface and numerical computation of the Euler equations, namely, the boundary value problem to determine a minimal surface and its conformally flat domain with a prescribed boundary configuration and an initial boundary value problem for the Euler equations on the minimal surface with the no-normal boundary condition.

In surface registration, it is crucial for computing differential equations on surfaces to choose a "good" parametrization and a parameter space. When we choose a uniformizing chart as the parametrization, we can represent every metric by $\lambda^2((dx^1)^2 + (dx^2)^2)$ for some positive function λ on the conformally flat domain which enables us to use the simplified coordinate representation of the differential equations as well as that of a geometric quantity such as the Gaussian curvature. Since the uniformizing chart is defined on the whole space of the surface, we can compute the differential equations on the domain without swapping charts as often as the domain which we focus on changes. From these reasons, let us numerically construct a uniformizing chart on a given surface. The normalized Ricci flow on a surface is helpful for our purpose. The normalized Ricci flow is defined as an evolution equation of Riemannian metrics and preserves the conformal structure and the area of an initial metric. Moreover, the solution of the normalized Ricci flow exponentially converges to a constant curvature metric in smooth topology. Hence we can obtain the conformally flat domain from the long-time limit of the solution. Ricci flow on not only a surface but also discrete one has been recently investigated and applied to numerical computation of a uniformizing chart. For general reference see [3]. In this talk, we provide a numerical scheme with high accuracy by using discrete Ricci flow and focusing on a geometric property of a minimal surface. Another key device in the scheme is the method of fundamental solution (MFS), which is a meshfree numerical solver for linear elliptic partial differential equations [4].

In numerical computation of the Euler equations, we adopt the vortex method as the numerical solver [5]. Thanks to the invariance of the vorticity along a fluid particles, discretizing an initial vorticity distribution with a liner combination of delta functions, called point vortices, we can treat the velocity field as a finite-dimensional Hamiltonian vector field whose Hamiltonian consists of a Green's function for Laplacian on the surface and the regularized Green's function by the geodesic distance, called Robin function. In particular, we examine dynamical evolution of point vortices on a minimal surface. In order to carry out theoretical analysis, assuming the existence of a non-trivial Killing vector field on a minimal surface, we provide exact solutions of the Hamiltonian system as an

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application of [6]. After introducing we introduce a numerical scheme for the Green's function and the Robin function by using MFS, we compare numerical results with the exact solutions. Finally, we investigate dynamical behavior of point vortices in terms of a boundary configuration and shape of a minimal surface by using the proposed numerical solver. This talk is based on a joint work with Dr. Koya Sakakibara (Kyoto University).

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ON INTERVAL DECOMPOSABILITY OF 2D PERSISTENCE MODULES

HIDETO ASASHIBA, MICKAËL BUCHET, EMERSON G. ESCOLAR, KEN NAKASHIMA AND MICHIO YOSHIWAKI

In persistent homology of filtrations, the indecomposable decompositions provide the persistence diagrams. In multidimensional persistence [1], it is known to be impossible to classify all indecomposable modules: There does not exist a complete discrete invariant that captures all the indecomposable modules. One direction is to consider the subclass of interval-decomposable persistence modules, which are direct sums of interval indecomposables. In this talk, we introduce the definition of pre-interval indecomposables, a more algebraic definition, and study the relationships among thin, pre-interval, and interval indecomposables.

Definition 0.1. The *equioriented 2D commutative grid* is the quiver

$$\begin{tikzcd}
\bullet \arrow[r] & \bullet \arrow[r] & \cdots \arrow[r] & \bullet \arrow[r] & \bullet \\
\bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \cdots \arrow[r] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[u] \\
\vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] & \vdots \arrow[u] \\
\bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \cdots \arrow[r] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[u]
\end{tikzcd}$$

with full commutative relation.

Then we show the follwoing statement over the equioriented 2D commutative grid.

Theorem 0.2. Let M be a indecomposable representation over the equioriented 2D commutative grid. Then the following are equivalent.

(1) M is thin,
(2) M is a pre-interval, and
(3) M is an interval.

Moreover, we provide an algorithm for answering the following question under certain finiteness conditions and without explicitly computing decompositions: Given an nD persistence module, determine whether or not it is (pre)interval-decomposable or thin-decomposable.

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WAISTS OF BALLS IN DIFFERENT SPACES

ARSENIY AKOPYAN

Gromov and Memarian [5, 6] have established the *waist inequality* asserting that for any continuous map $f : \mathbb{S}^n \to \mathbb{R}^{n-k}$ there exists a fiber $f^{-1}(y)$ such that every its t-neighborhood has measure at least the measure of the t-neighborhood of an equatorial subsphere $\mathbb{S}^k \subset \mathbb{S}^n$.

Going to the limit we may say that the $(n-k)$-volume of the fiber $f^{-1}(y)$ is at least that of the standard subsphere $\mathbb{S}^k \subset \mathbb{S}^n$. We extend this limit statement to the exact bounds for balls in spaces of constant curvature, tori, parallelepipeds, projective spaces and other metric spaces.

By the volume of preimages for a non-regular map f we mean its *lower Minkowski content*, some new properties of which will be also presented in the talk.

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EVERY 1D PERSISTENCE MODULE IS A RESTRICTION OF SOME INDECOMPOSABLE 2D PERSISTENCE MODULE

MICKAËL BUCHET AND EMERSON G. ESCOLAR

A recent work by Lesnick and Wright [1] proposed a visualisation of 2D persistence modules by using their restrictions onto lines, giving a familty of 1D persistence modules. We explore what 1D persistence modules can be obtained as a restriction of indecomposable 2D persistence modules to a single line. To this end, we give a constructive proof that any 1D persistence module can in fact be found as a restriction of some indecomposable 2D persistence module. As another consequence of our construction, we are able to exhibit indecomposable 2D persistence modules whose support has holes.

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A LIMIT THEOREM FOR PERSISTENCE DIAGRAMS OF RANDOM COMPLEXES BUILT OVER MARKED POINT PROCESSES

KIYOTAKA SUZAKI

A persistence diagram is an expression of a persistent homology, which is an important tool to understand topological features (connected components, rings, cavities, etc) of data. A standard way to convert input data into a filtered simplicial complex with parameter $t \geq 0$ is to use the Čech complexes, i.e., the family of nerves of the t-balls centered at each data point.

In this talk, a filtration of simplicial complexes is constructed from finite marked data points in Euclidean space. Examples of our construction include a family of nerves of sets with various sizes, growths, and shapes. In addition, we consider the case when input data are marked point processes (randomly distributed marked points). We then discuss a strong law of large numbers of these persistence diagrams as the size of the window observing random data tends to infinity.

This talk is based on a joint work with Tomoyuki Shirai (Kyushu University).

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A DERIVED ISOMETRY THEOREM FOR CONSTRUCTIBLE SHEAVES ON $\mathbb{R}$

NICOLAS BERKOUK (JOINT WORK WITH GRÉGORY GINOT)

Persistent homology has been recently studied with the tools of sheaf theory in the derived setting by Kashiwara and Schapira [KS18a] after J. Curry has made the first link between persistent homology and sheaves.

We prove the isometry theorem in this derived setting, thus expressing the convolution distance of sheaves as a matching distance between combinatorial objects associated to them that we call graded barcodes. This allows to consider sheaf-theoretical constructions as combinatorial, stable topological descriptors of data, and generalizes the situation of persistence with one parameter.

On a second time, we relate sheaf-theoretic and persistence-theoric constructions, and show how the derived isometry theorem allow to give a new, deeper, interpretation of level-set persistence stability.

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Synaptic plasticity through topological methods

Synaptic plasticity is defined as the variation in strength of synaptic connection between neurons as well as the creation of new connections and elimination of existing connections. The brain is capable of making such changes as a reaction to stimuli and the process is considered by neuroscientists as one of the fundamental aspects of learning and other highly sophisticated and delicate brain functions. This talks is a report on work in progress with a team of mathematicians and scientists from the Blue Brain Project, on the application of topological tools to a digital reconstruction of a small section of the brain that is capable of simulating synaptic plasticity.

LOCALIZATION OF THE NEURAL CURRENT SOURCE IN THE HUMAN BRAIN BASED ON A MAPPING FROM A SPHERE TO THE CORTICAL SURFACE

TAKAAKI NARA AND KENTA KABASHIMA

In this talk, we propose a novel method for Magnetoencephalography (MEG) inverse problems in which the neural current source inside the human brain is identified from the measured magnetic field outside the head. The conventional approaches to this inverse problem are categorized into two groups: parametric methods and imaging approaches. The former assumes that the current source is expressed by a finite number of equivalent current dipoles, and reconstructs its number, positions, and moments via the non-linear least squares method. The latter assumes that the current source is fixed on grids on a cortical surface and solves for their moments. However, the problems are that the former method cannot identify the spatial extent of sources, whereas the latter obtains too smoothed solution by L2 regularization or too focal solution by L1 regularization.

To this problem, we propose a novel parametric approach to identify a source domain with spatial extent by using a mapping from a sphere to the cortical surface [1]. We express a source region on the cortical surface as a domain mapped from a circle on the sphere. As a result, a single source domain on the cortical surface can be represented by three parameters: the center (θ_0, ϕ_0) and radius r_0 of the circle on the sphere. Then, we minimize a squared error between the measured data and the theoretical magnetic field represented by those parameters. Since the parameters can be assumed in a Cartesian product set, we can apply an optimization algorithm based on the Lipschizian continuity that efficiently obtains a global minimum [2]. In this way, the neural current source domain with spatial extent can be parametrically identified. After verifying the proposed method with numerical simulations, real data analyses will be shown.

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CONLEY COMPLEXES AND CONNECTION MATRICES IN COMBINATORIAL TOPOLOGICAL DYNAMICS

MARIAN MROZEK AND THOMAS WANNER

1. Introduction

Connection matrices have been introduced by R. Franzosa [7] as an algebraic topological tool in the study of Morse decompositions of flows on locally compact metric spaces. As observed by Robbin and Salamon [11], in the setting of field coefficients the algebraic part of the construction of connection matrix may be decoupled from the dynamical part by defining connection matrices for lattice filtered chain complexes and applying this general concept to the lattice of attracting neighboorhoods. Harker, Mischaikow and Spendlove [8] expand these ideas by introducing what they call a Conley complex of a poset-graded chain complex or lattice-filtered chain complex. This is a poset graded chain complex chain homotopic to the given one whose boundary map vanishes on the diagonal. They prove that Conley complex is unique up to a chain graded isomorphism. They define the connection matrix of a poset-graded chain complex or lattice-filtered chain complex as the boundary operator of a Conley complex. Since chain isomorphic complexes may differ in their boundary operators, the connection matrix need not be unique despite the fact that Conley complex is unique up to isomorphism.

In this note we apply the ideas of [11, 8] to define connection matrices for Morse decompositions of combinatorial multivector fields [10], an extension of Forman's combinatorial vector fields [5, 6]. Combinatorial multivector fields may be constructed from clouds of vectors [10, 2]; hence, they constitute a natural tool to analyze and classify dynamical data. The importance of connection matrices in this context, similarly to the case of flows, lies in the fact that a non-zero entry in the connection matrix implies the existence of a heteroclinic connection between the respective Morse sets. Moreover, it is natural to expect that the Conley complex may be helpful in classifying dynamical data.

We present an example that also in the combinatorial setting connection matrices need not be unique. But, we prove that they are unique in the case of Morse decomposition of a gradient combinatorial vector field. We also indicate some relations between persistence [3], combinatorial vector fields [5] and Conley complexes [8].

2. Main result

A Lefschetz complex (see [10] for the definition), originally defined by S. Lefschetz and called a cell complex in [8], is an abstraction of a finite combinatorial complex such as simplicial complex or cubical complex. A Lefschetz complex consists of a set of cells X and a map κ which assigns to every pair of cells a ring element called incidence coefficient. The incidence coefficient encodes the face relation between cells. Cells constitute a natural basis of the associated chain complex $C(X)$ with boundary operator defined in terms of the incidence coefficents. In this note we assume that incidence coefficients are from a fixed field $\mathbb{F}$.

A remarkable feature of every Lefschetz complex is that the face relation in X induces a T_0 Alexandrov topology $\mathcal{T}_X$ on X. This makes every Lefschetz complex X a finite topological space $(X, \mathcal{T}_X)$.

A combinatorial multivector field $\mathcal{V}$ on a Lefschetz complex X, originally defined in [10] and in this note considered in a weaker version introduced in [2] (see also [9]), is a partition of X into non-empty, locally closed sets (see [4, Sec. 2.7.1, pg 112]) in the topology $\mathcal{T}_X$. The elements of the partition are called *multivectors*. A multivector is called a *vector* if it has no more than two elements.

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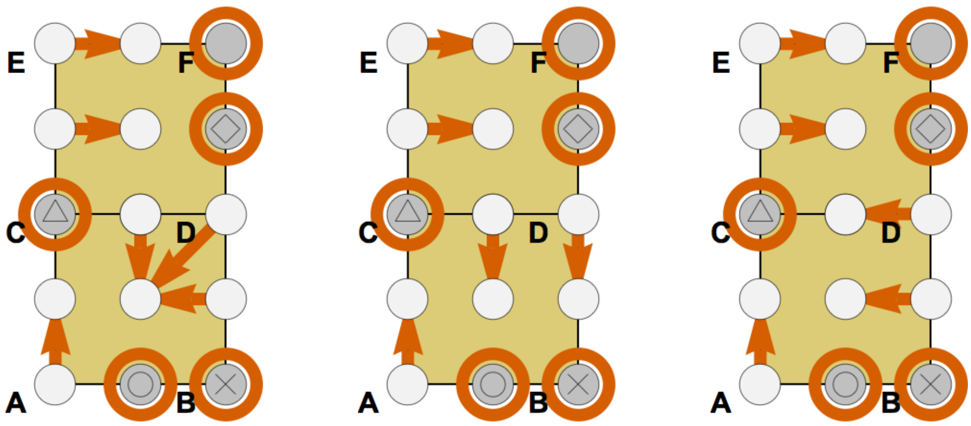

FIGURE 1. A multivector field (left) and its two combinatorial vector fields (middle and right).

In this case it has the form $V = \{V^-, V^+\}$ where either $V^- = V^+$ or V^- is a face of V^+ of codimension one.

A combinatorial multivector field $\mathcal{V}$ on a Lefschetz complex X induces a dynamical system on X. This, in particular, means that one can define isolated invariant sets, attractors, repellers and Morse decompostions [10]. For each Morse decomposition $\mathcal{M}$ there is a lattice of attracting neighbourhoods which induces a lattice filtered chain complex. In particular, one can associate with $\mathcal{M}$ the Conley complex and a non-empty collection of connection matrices. As we show in the next section, the connection matrix need not be unique. But, we prove the following theorem.

Theorem 2.1. Assume $\mathcal{V}$ is a gradient combinatorial vector field on a Lefschetz complex X. Then, the Morse decomposition consisting of all the critical cells of $\mathcal{V}$ has precisely one connection matrix. It coincides with the matrix of the boundary operator of the associated Morse complex.

3. AN EXAMPLE

Three examples of a combinatorial multivector field are presented in in Figure 1. The middle and right example are actually combinatorial vector fields, since there are no multivectors of cardinality greater than two. All three examples have the same collection of critical cells $\mathcal{M} := \{B, C, F, AB, DF\}$ and $\mathcal{M}$ is a Morse decomposition for all of them. One can verify that the left example has two connection matrices with coefficients in $\mathbb{Z}_2$:

$$C_1 := \begin{array}{c|ccccc} & B & C & F & AB & DF \\ \hline B & 0 & 0 & 0 & 1 & 1 \\ C & 0 & 0 & 0 & 1 & 0 \\ F & 0 & 0 & 0 & 0 & 1 \\ AB & 0 & 0 & 0 & 0 & 0 \\ DF & 0 & 0 & 0 & 0 & 0 \end{array} \quad \text{and} \quad C_2 := \begin{array}{c|ccccc} & B & C & F & AB & DF \\ \hline B & 0 & 0 & 0 & 1 & 0 \\ C & 0 & 0 & 0 & 1 & 1 \\ F & 0 & 0 & 0 & 0 & 1 \\ AB & 0 & 0 & 0 & 0 & 0 \\ DF & 0 & 0 & 0 & 0 & 0 \end{array}\ .$$

Hence, as in the case of classical dynamical systems, connection matrices in the combinatorial setting need not be unique. However, as Theorem 2.1 implies, matrix C_1 is the unique matrix of the combinatorial multivector field in the middle and matrix C_2 is the unique matrix of the combinatorial multivector field in the right of Figure 1. Note that there are examples that the connection matrix need not be unique also for non-gradient combinatorial vector fields.

4. RELATION TO PERSISTENCE.

It is known that homological persistence [3] may be phrased in terms of combinatorial Morse theory [1]. This observation may be extended to Conley complexes as follows. Assume that $\mathcal{X} = \{X_0, X_1, \ldots X_n\}$ is a filtration of a Lefschetz complex X, that is $\emptyset = X_0 \subset X_1 \subset \cdots \subset X_n = X$ is a tower of $\mathcal{T}_X$-closed subcomplexes of X. For each $x \in X$ let $t(x) := \min\{\, i \mid x \in X_i \,\}$ denote the time of appearance of x in the filtration $\mathcal{X}$. Denote by $D(\mathcal{X})$ the persistence diagram of the associated filtration of chain complexes $0 = C(X_0) \subset C(X_1) \subset \cdots \subset C(X_n)$. Recall that the persistence diagram is a multiset consisting of pairs (p, q) where p is the birth time of a homology class and q is its death time or infinity if the class never dies.

We say that a combinatorial vector field $\mathcal{V}$ on X is a *persistence combinatorial vector field* with respect to the filtration $\mathcal{X}$ if the map $\alpha : \mathcal{V} \to D(\mathcal{X})$ given by

$$\alpha(V) := \begin{cases} (t(V^-), t(V^+)) & \text{if } V^- \neq V^+, \\ (t(V^-), \infty) & \text{if } V^- = V^+ \end{cases}$$

is a bijection of multisets.

The filtration $\mathcal{X}$ is obviously a lattice with respect to union and intersection. This makes $C(X)$ a filtered chain complex and allows one to associate with $\mathcal{X}$ a Conley complex $\mathrm{Con}(\mathcal{X})$.

Theorem 4.1. Given a filtration $\mathcal{X}$ of a Lefschetz complex X there is another Lefschetz complex $\bar{X}$ and a bijection $\theta : X \ni x \mapsto \bar{x} \in \bar{X}$ such that

(i) $\bar{\mathcal{X}} := \{\theta(X_0), \theta(X_1), \ldots, \theta(X_n)\}$ is a filtration of $\bar{X}$,
(ii) θ induces a chain isomorphism of filtered chain complexes $C(X)$ and $C(\bar{X})$,
(iii) $\bar{X}$ admits a persistence combinatorial vector field with respect to $\bar{\mathcal{X}}$,
(iv) the Conley complexes of $\mathcal{X}$ and $\bar{\mathcal{X}}$ coincide,
(v) in particular, persistence diagrams of $\mathcal{X}$ and $\bar{\mathcal{X}}$ coincide.

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HELIOSEISMIC AND MAGNETIC IMAGER DATA CLASSIFICATION USING COMBINATORIAL TOPOLOGICAL DYNAMICS

MATEUSZ JUDA

1. Introduction

In this note we present a method for topological features extraction of sampled vector fields. By a sampled vector field we mean a finite set of points in $\mathbb{R}^d$ with vectors attached. Such data arise in a natural way from sampling dynamics. As a real world example we study the data collected by the Helioseismic and Magnetic Imager (HMI) - an instrument designed to study the magnetic field on the surface of Sun [4]. We show that the proposed method significantly outperforms the presently available methods in the HMI solar flare classification task. Our method is general and can be applied to any sampled vector field data, however in this work we present results based only on HMI data.

This note is based on research projects with: Marian Mrozek, Bartosz Zielinski, Tomasz Kapela, Matthias Zeppelzauer.

2. HMI data

The goal of HMI project is to study the relationship between the behavior of the photospheric magnetic field and solar activity. In particular, space weather anomalies are linked to solar flares - a sudden explosion of energy. Solar flares can interfere with satellites and also with equipment such as power utility grids, electronics etc. Predicting solar flares is a challenging task. The recent prediction techniques are based on machine learning (ML) methods. Typically, ML methods for solar flares prediction use 25 numerical characteristics of the magnetic field, the so called data features: total unsigned current helicity, total magnitude of Lorentz force etc.

3. Methodology

We propose to extract features of a sampled vector field using a method based on combinatorial multivector fields [5], a generalization of Forman's combinatorial vector fields [9, 8]. Namely, as a first step we reconstruct dynamics given by a cloud of vectors by building a simplicial complex $\mathcal{K}$ on the point cloud and a combinatorial multivector field $\mathcal{V}$ on $\mathcal{K}$. This way we obtain a graph on the set of all simplices with edges approximating the vector field. We analyze a collection of such graphs using DeepWalk [2] approach which transforms graphs into text documents. Next we use Fasttext [1] to learn embedding of words into $\mathbb{R}^d$, where d is a fixed parameter. Using that embedding we get a representation of the text documents in $\mathbb{R}^d$. The representation gives us a feature vector for each sampled vector field. In the following sections we present more details of the method.

3.1. **Multivector fields.** By a *combinatorial dynamical system* on a simplicial complex K (*cds* in short) we mean a multivalued map $F : K \multimap K$, that is a map which sends each simplex in K into a family of simplices in K. The cds F may be viewed as a digraph G_F whose vertices are simplices in K with a directed edge from simplex σ to simplex τ if and only if $\tau \in F(\sigma)$. However, F is more than just the digraph G_F because K, the set of vertices of G_F, is a finite topological space with Alexandrov topology given by the poset of face relation [11].

We construct a cds from a cloud of vectors in two steps. In the first step the cloud of vectors is transformed into a combinatorial multivector field [5]. In the second step, the combinatorial multivector field is transformed into a cds. In order to explain the steps, we introduce some definitions. We say that $A \subset K$ is *convex* if for any $\sigma_1, \sigma_2 \in A$ and $\tau \in K$ such that σ_1 is a face of τ and τ is a face of

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σ_2 we have $\tau \in A$. We note that convex subsets of K are precisely the locally closed sets of K (see [6, Sec. 2.7.1, pg 112]) in the Alexandrov topology of K. We define a *multivector* as a convex subset of K and a *combinatorial multivector field* on K (*cmf* in short) as a partition $\mathcal{V}$ of K into multivectors. Given a cmf $\mathcal{V}$, we denote by $[\sigma]_{\mathcal{V}}$ the unique V in $\mathcal{V}$ such that $\sigma \in V$. We associate with $\mathcal{V}$ a cds $F_{\mathcal{V}} : K \multimap K$ given by $F_{\mathcal{V}}(\sigma) := \operatorname{cl}\sigma \cup [\sigma]_{\mathcal{V}}$.

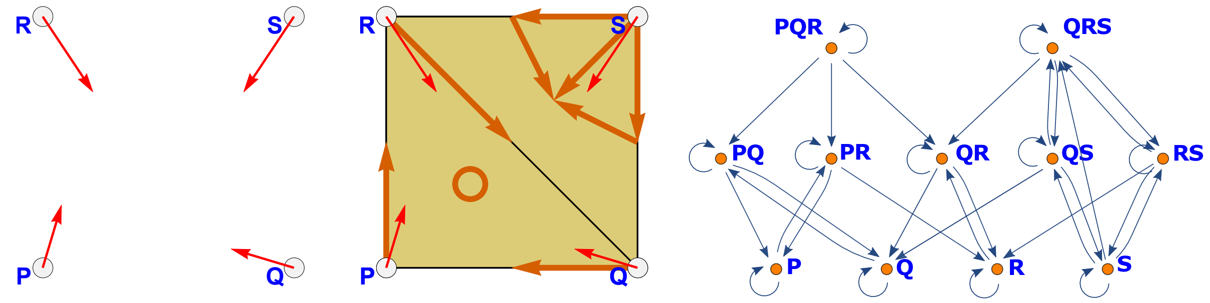

FIGURE 1. Left: A cloud of vectors. Middle: A possible combinatorial multivector field representation of the cloud of vectors. Right: The associated combinatorial dynamical system represented as a digraph.

Figure 1(left) presents a toy example of a cloud of vectors. It consists of four vectors marked red at four points **P**, **Q**, **R**, **S**. One of possible geometric simplicial complexes with vertices at points **P**, **Q**, **R**, **S** is the simplicial complex K consisting of triangles **PQR**, **QRS** and its faces. A possible multivector field $\mathcal{V}$ on K constructed from the cloud of vectors consists of multivectors $\{\mathbf{P}, \mathbf{PR}\}$, $\{\mathbf{R}, \mathbf{QR}\}$, $\{\mathbf{Q}, \mathbf{PQ}\}$, $\{\mathbf{PQR}\}$,$\{\mathbf{S}, \mathbf{RS}, \mathbf{QS}, \mathbf{QRS}\}$. It is indicated in Figure 1(middle) by orange arrows between centers of mass of simplices. Note that in order to keep the figure legible, only arrows in the direction increasing the dimension are marked. The singleton $\{\mathbf{PQR}\}$ is marked with an orange circle. The associated combinatorial dynamical system $F_{\mathcal{V}}$ presented as a digraph is in Figure 1(right). Note that in general K and $\mathcal{V}$ are not uniquely determined by the cloud of vectors.

We denote by $G_{\mathcal{V}}$ the graph obtained from G_F by contracting to a point the vertices in G_F sharing the same multivector.

3.2. **DeepWalk.** In order to analyze a collection of graphs $G_{\mathcal{V}}$ we use DeepWalk [2]. The method is used to analyze graphs as text documents with Natural Language Processing (NLP). Given $G_{\mathcal{V}}$ we generate a set of paths, that is random walks of length not exceeding a fixed k. We assume that for each vertex a word from a vocabulary is given as the vertex label. For a path p we generate a sentence by replacing each vertex on p by its label. A set of such sentences constitutes a text document associated with the set of paths. In this context the order of sentences is not important. For a given set of graphs we consider the documents as a text corpus. Using NLP techniques, in particular Fasttext [1], we learn the representation of words as vectors in $\mathbb{R}^d$ with a fixed d. Each document is represented as the average of its word vectors.

3.3. **Topological vocabulary.** The NLP procedure described above requires a vocabulary in order to assign labels to the vertices. We construct labels which graspe some local, topological properties of the vertex in the vector field. More precisely given a multivector $V \in \mathcal{V}$, that is a vertex in $G_{\mathcal{V}}$, we first define the *label of V at level 0*, denoted $l_0(V)$, as a tuple

$$l_0(V) := (\max_{\sigma\in V} \dim\sigma, |V|, \chi(V)),$$

where $\dim\sigma$ denotes the dimension of simplex σ, $|V|$ stands for the cardinality of V, and $\chi(V)$ is the Euler characteristic of V. We define *label of V at level d* , denoted $l_d(V)$, as a tuple

$$l_d(V) := (l_0(V), \operatorname{sorted}(\{l_0(u) \mid u \in N_d^+(V)\}), \operatorname{sorted}(\{l_0(u) \mid u \in N_d^-(V)\})),$$

where $N_d^+(V)$ (resp. $N_d^-(V)$) are sets of vertices in the forward (resp. backward) distance from V not bigger than d.

As an example we consider the multivector field and the graph G_F presented in Figure 1. Figure 2 presents the associated graph on multivectors $G_{\mathcal{V}}$. Table 1 presents step by step calculations of the labels at level 1.

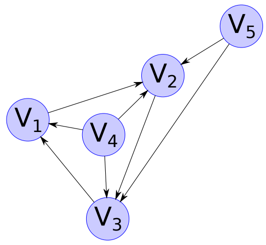

FIGURE 2. $G_{\mathcal{V}}$ graph for the example presented in Figure 1.

V	simplices of V	$l_0(V)$	$N_1^+(V)$	$N_1^-(V)$	$l_1(V)$
V_1	$\{\mathbf{P}, \mathbf{PR}\}$	$(1,2,0)$	$\{V_2\}$	$\{V_3, V_4\}$	$((1,2,0), [(1,2,0)], [(1,2,0), (2,1,1)])$
V_2	$\{\mathbf{R}, \mathbf{QR}\}$	$(1,2,0)$	$\{V_3\}$	$\{V_1, V_4, V_5\}$	$((1,2,0), [(1,2,0)], [(1,2,0), (2,1,1), (2,4,0)])$
V_3	$\{\mathbf{Q}, \mathbf{QP}\}$	$(1,2,0)$	$\{V_1\}$	$\{V_2, V_4, V_5\}$	$((1,2,0), [(1,2,0)], [(1,2,0), (2,1,1), (2,4,0)])$
V_4	$\{\mathbf{PQR}\}$	$(2,1,1)$	$\{V_1, V_2, V_3\}$	$\emptyset$	$((2,1,1), [(1,2,0), (1,2,0), (1,2,0)], \emptyset)$
V_5	$\{\mathbf{S}, \mathbf{RS}, \mathbf{QS}, \mathbf{QRS}\}$	$(2,4,0)$	$\{V_2, V_3\}$	$\emptyset$	$((2,4,0), [(1,2,0), (1,2,0)], \emptyset)$

TABLE 1. Step by step calculation of labels at level 1 for the example presented in Figure 1 and Figure 2

4. RESULTS

To evaluate our method we use a data set proposed in [3]. The data set provides 823 HMI magnetograms. The state-of-the-art methods extract from each magnetogram 13 real number characteristics. Additionally, for each magnetogram we know a flare class (B, C, M, and X) according to the maximum magnitude of flares generated in the approaching 24 hours. Our goal is to find an ML model for the flare class prediction based on the megnetograms.

We use randomly selected 70% of the data as a training set, and the rest as a test set. We transform the magnetograms from the training set into text documents and create a model of the artificial language described above. Then, for each magnetogram (training and test), we create a new feature vector using the word embeddings. We present results obtained with the following parameters:

- level of labels is $k = 4$;
- for each label l we select randomly 50% of vertices v in $G_{\mathcal{V}}$, such that $l_k(v) = l$;
- for each selected vertex we generate a random walk which begins at v and a random walk which ends at v, both of length 20;
- the dimension of the word embeddings is 40.

To compare the state-of-the-art feature vector with the new one we compare classification metrics for LinearSVC [7] and AdaBoostClassifier [10] from sklearn python library. We provide classiffiers scores in Table 2. We observe that the features based on the proposed word embeddings always are significantly better than the state-of-the-art features. We emphesize that the proposed method outperforms state-of-the-art for the test set.

	Classifier	test	training
proposed feature vector	LinearSVC	0.898	0.881
	AdaBoostClassifier	0.846	0.994
state-of-the-art feature vector	LinearSVC	0.417	0.392
	AdaBoostClassifier	0.663	0.918

TABLE 2. Classifiers scores on test and training data sets.

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INTEGRABLE BILLIARDS: GENERALIZATIONS AND APPLICATIONS TO MECHANICS

VICTORIA V. VEDYUSHKINA

Let us recall that billiard system describes motion of a particle in a flat domain Ω with piecewise smooth boundary P. Reflection should be elastic. The Hamiltonian is the square of velocity vector.

D.Birkhoff proved the integrability if P is an ellipse. V.V.Kozlov, D.V.Treschev proved that integrability preserves for P that consists of arcs of confocal ellipses and hyperbolas. This system has an additional first integral Λ which value is some parameter of the caustic for trajectory.

Fixing $|\overrightarrow{v}|^2 = h$ one have 3-dimensional manifold Q^3 foliated on level surfaces of Λ. Such foliations are smooth-wise analogs of Liouville foliations investigated by A.T. Fomenko school. Fomenko–Zieschang invariant (graph with numerical marks, vertices correspond to singularities of the foliation) classifies them in the sense of Liouville equivalence. Two integrable systems are called equivalent if piece-wise diffeomorphism exists. Their trajectory closures also have the same structure.

Fomenko–Zieschang invariant for billiards in flat domains were calculated by V. Dragovich, M. Radnovich and V.V. Vedyushkina (Fokicheva). Let us call such plane domains elementary domains.

Now we describe a generalization of such billiard. Let us glue together several elementary domains-sheets along common borders. Produced domain has a structure of CW-complex. Let us call it a *billiard book*. An interesting problem is to describe the Liouville foliation of the obtained billiards.

Previously, the case of only two glued domains-sheets was considered. Produced domains were called *topological billiards*. They were completely investigated in terms of the Fomenko-Zieschang invariant in [3]. Later these invariants were calculated for wide class of non-trivial billiard books.

On the other hand Fomenko-Zieschang invariants were calculated for many integrable cases of the rigid body dynamics and geodesic flows. It allows to detect the Liouville equivalence of these systems to some topological billiards by comparing the invariants (see [2]). It means that billiard books and topological billiards "visually model" many fairly complicated integrable cases in the dynamics of the rigid body. This simulation makes it possible to present and effectively classify the stable and unstable periodic trajectories of integrable systems, in particular, in physics and mechanics.

For example, the Euler case can be simulated by the billiards for all values of energy integral. Such billiard simulation is done for the systems of the Lagrange top and Kovalevskaya top, then for the Zhukovskii gyrostat, for the systems by Goryachev-Chaplygin -Sretenskii, Clebsch, Sokolov, the Kovalevskaya-Yahia case (Kovalevskaya top with gyrostat) for many values of energy.

Also it was possible to apply results of our calculation to modeling integrable geodesic flows on orientable 2-dimensional surfaces. Namely, all such flows that have linear or quadratic first integral were modeled by integrable billiards. It means that every linear or quadratic integral can be realized in this sense by one, canonical Hamiltonian and quadratic first integral.

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A COMPUTATIONAL FRAMEWORK FOR CONNECTION MATRIX THEORY

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ABSTRACT. Algebraic topology and dynamical systems are intimately related: the algebra may constrain or force the existence of certain dynamics. Morse homology is the prototypical theory grounded in this observation. Conley theory is a far-reaching topological generalization of Morse theory and the last few decades have seen the development of a computational version of the Conley theory. The computational Conley theory is a blend of combinatorics, order theory and algebraic topology and has proven effective in tackling problems within dynamical systems.

Within the Conley theory the connection matrix is the mathematical object which transforms the approach into a truly homological theory; it is the Conley-theoretic generalization of the Morse boundary operator. We'll discuss a new formulation of the connection matrix theory, which casts the connection matrix in categorical, homotopy-theoretic language. This enables the efficient computation of connection matrices via the technique of reductions in combination with algebraic-discrete Morse theory. We will also discuss a software package for such computations. Time permitting, we'll demonstrate our techniques with an application of the theory and software to the setting of transversality models [9]. This application allows us to compute connection matrices for the classical examples of Franzosa [5] and Reineck [13] as well as high-dimensional examples from a Morse theory on spaces of braid diagrams introduced in [6].

INTRODUCTION

Topology and algebraic invariants have played a prolific role in dynamical systems [1, 16]. Loosely stated, a dynamical system engenders topological data: both local (e.g. fixed points) and global (e.g. attractors). The topological data have associated algebraic invariants (e.g. homology) and the relationship between local and global is codified in the algebra.

Morse theory is an influential instantiation of this idea wherein the local data (nondegenerate fixed points) in the gradient flow $\dot{x}(t) = -\nabla f(x(t))$ of generic map $f : M \to \mathbb{R}$ are graded by their Morse index and contribute to a chain complex $(C_\bullet, \partial)$. The boundary operator is determined by the structure of the connecting orbits. It is classical that the Morse homology $H_\bullet(C_\bullet, \partial)$ is isomorphic to the singular homology $H_\bullet(M)$. Conley theory is a purely topological generalization of Morse theory: the index of an isolated invariant set is a topological space whose homology gives a coarse description of the unstable dynamics. Essential to the Conley index is the property of *continuation*: the index is robust to perturbations of the system [1]. Our recent work [7] concerns developing a categorical, homotopy-theoretic framework for the computation of connection matrices, the Conley-theoretic generalization of the Morse boundary operator [5]. We outline a computational connection matrix theory and give application to transversality models in [9]. Moreover in [8] we give the specifics on the particulars of the algorithm, including a novel scheme for an implicit discrete Morse theory on cubical complexes.

COMPUTATION OF CONNECTION MATRICES

Analogous to the Morse boundary operator, the *connection matrix* is a boundary operator defined on Conley indices [5]. In contrast to the Morse boundary operator, the connection matrix is not obtained directly from the trajectories, but it is related to them. This relationship implies the basic utility of a connection matrix is to prove existence of connecting orbits [10]. At a higher level, it serves as an algebraic representation of global dynamics and may used in some cases to construct semi-conjugacies of the global attractor [3, 12]. Ultimately, the connection matrix completes the Conley theory to a homological theory [11] for dynamical systems.

In recent work [7] we gave a categorical, homotopy-theoretic treatment of the connection matrix theory. In this setting we can interpret a connection matrix as the boundary operator of a particularly simple representative of an isomorphism class in an appropriate homotopy category. We show that, in the case of fields, the use of homotopy categories enables the connection matrix theory to be made functorial.

Using the homotopy-theoretic framework, the computation of a connection matrix can be phrased in terms of (filtered) reductions, a technique introduced in [4] and extensively used in [15]. In the case of the computational Conley theory, where the typical input is a decomposition of a cell complex into attracting blocks, we show that discrete Morse theory induces a reduction and can be used to provide an efficient algorithm for computing connection matrices. This provides a purely algorithmic and constructive proof of existence of connection matrices [5, 14]. Moreover, we'll discuss publicly available software packages for the connection matrix theory [2].

Application to a Morse Theory on Braids

As developed in [6], a set $\{u^n(t,x)\}$ of solutions to a scalar parabolic partial differential equation of the form $u_t = u_{xx} + f(u_x, u, x)$ may be lifted to (x, u, u_x)-space to create a braid. The space of braids partitions into isotopy braid classes and monotonicity properties of the PDE induce dynamics on braid classes. Discretized braids (whose strands are piecewise linear) are a finite-dimensional approximation to the space of braids. In this case the phase space partitions into a cubical complex of discrete braid classes. The parabolic PDE induces dynamics on braid classes via the comparison principle, which leads to the notion of a Conley index for braid classes. We will discuss applications of the algorithms to compute connection matrices in this setting [9], including examples of 10 and 12-dimensional cubical complexes. The insights obtained from these high-dimensional computations have led to new conjectures for the theory [9].

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SUB-IMAGE ANALYSIS USING TOPOLOGIAL SUMMARY STATISTICS

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We propose a Sub-Image aNAlysis using Topological summaRy stAtistics (SINATRA) framework for pipelining image analysis, with the aim of understanding what differences in shapes constitute to changes in regression outcomes. The pipeline consists of four steps. The first step is to transform the shapes into functions with the Euler Characteristic Transformation (ECT)[1]. This makes the shapes amenable to the tools of functional data analysis. The second step is to fit a Bayesian Gaussian process classification model on the transformed shapes. The third step is to assign a non-linear, Kullback-Leibler divergence based importance metric called Relative Centrality (RATE)[2] to the classification model. The importance metric allows us to do association mapping to perform feature selection on the Gaussian regression model. Finally, we devise a partial inverse Euler Characteristic Transformation formula inspired by the finite injectivity result of ECT proved in [3]. We use the inversion formula to pull back the selected features to the shapes, which allows us to infer what features of the shape were associated with the classification decision. The main contributions of this work are the integration of steps one to four to a image-analysis pipeline, and the discretization of the theoretically satisfying results to drive concrete applications.

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ANALYZING SPHERE PACKINGS WITH HIGHER ORDER PERSISTENCE.

GEORG OSANG

Persistent homology has become a popular tool to analyse various kinds of data, in particular in material sciences. Specifically, persistence of discrete point sets has recently been used to analyse sphere packing data, to shed light on structures arising in sphere packings at different packing densities. [1] We generalize this notion and introduce higher-order persistence of discrete point sets. [2] We briefly address computational challenges, and then show how this notion can deal with noisy point samples. In the setting of sphere packings we show that this notion can also capture a wider variety of local structures, and in particular can distinguish between the hexagonal close packing and the face centered cubic lattice packing, two structures know to have optimal packing density in 3 dimensions.

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MANIPULATING HOLE SYSTEMS

KATHARINA ÖLSBÖCK

We want to reconstruct the shape of a point cloud, with focus on the holes of the resulting model. In many cases, the Alpha complex of appropriate scale gives a good reconstruction. However, in some applications the holes of the model are important and there is no scale of the Alpha complex that gives a satisfactory result. We define operations to change the birth and death of holes in a filtered simplicial complex, i.e., they open or close holes in a subcomplex of a fixed scale. Manipulating a hole can have side effects on other holes. We study the dependences between holes, which enables us to predict or counteract these side effects.
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CONVEX FAIR PARTITIONS INTO ARBITRARY NUMBER OF PIECES

S. AVVAKUMOV,
JOINT WORK WITH A. AKOPYAN AND R. KARASEV

In [4] a very natural problem was posed: Given a positive integer m and a convex body K in the plane, cut K into m convex pieces of equal areas and perimeters.

The case $m = 2$ of the problem is done with a simple continuity argument. The case $m = 2^k$ could be done similarly using the Borsuk–Ulam–type lemma by Gromov [5]. Further cases, $m = p^k$ for a prime p, were established in [3] and [2] independently.

In the talk I will outline the proof for arbitrary m which was recently obtained in [1]. We will see how equivariant obstruction method was used to solve the special cases of the problem, why it failed in the general case, and what new ideas were required to move further.

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RANK INVARIANT FOR ZIGZAG MODULES

WOOJIN KIM AND FACUNDO MÉMOLI

The rank invariant [2] is of great interest in studying standard persistence modules over $\mathbb{R}^n$ [3, 4, 5, 6, 7]. In particular, it is well known that

(a) the rank invariant is a complete invariant for (standard) persistence modules over $\mathbb{R}$, and
(b) for $n \geq 2$ the rank invariant fails to be complete for (standard) persistence modules over $\mathbb{R}^n$.

Motivated by these facts, in our work, we wondered whether it would be possible to define a suitable notion of rank invariant for *zigzag* modules, and whether the resulting invariant would be complete.

We elucidate such a generalization of the rank invariant to zigzag modules by establishing a certain relationship between the limits and colimits of every subdiagram of a zigzag module. Following the formulation of [1], we consider zigzag modules represented over $\mathbb{R}$. Then, given a zigzag module M over $\mathbb{R}$, for every $s \leq t$ in $\mathbb{R}$, we study the canonically induced map from the limit to the colimit of the subdiagram $M|_{[s,t]}$:

$$\phi_M(s,t) : \varprojlim M|_{[s,t]} \longrightarrow \varinjlim M|_{[s,t]}.$$

Then, the rank invariant function $\mathrm{rk}(M) : \{(s,t) \in \mathbb{R}^2 : s \leq t\} \to \mathbb{Z}$ associated to M is defined by

$$\mathrm{rk}(M)(s,t) := \mathrm{rank}\left(\phi_M(s,t)\right).$$

In particular, we prove that the rank invariant function of a zigzag module recovers its interval decomposition. Our results therefore imply that *our rank invariant is a complete invariant for zigzag modules.*

Complementing the characterization result mentioned above, we also show that the erosion distance (as in the work of A. Patel [6]) between the rank invariant functions associated to two arbitrary zigzag modules is bounded from above by the interleaving distance [1] between the zigzag modules (up to a multiplicative constant).

As a further extension, our construction allows us to extend the notion of generalized persistence diagram by A. Patel [6] to zigzag modules valued in any symmetric monoidal bicomplete category.

NOTE: A preprint with a full description of these ideas is available in our recent arxiv preprint `https://arxiv.org/abs/1810.11517`.

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FOMENKO–ZIESCHANG INVARIANTS AND TOPOLOGY OF KOVALEVSKAYA INTEGRABLE SYSTEMS

VLADISLAV A. KIBKALO

This talk will be devoted to topological invariants that classify foliations of integrable Hamiltonian systems [1]. They will be applied to describe closures of trajectories and topology of invariant submanifolds for integrable analogs of Kovalevskaya case in rigid body dynamics.

Let us recall that a system $v = \mathrm{sgrad}\, H$ with 2 degrees of freedom is integrable (in Liouville sense) if it has a first integral K that is independent of the Hamiltonian H (the energy integral). The phase space is foliated on 2-dimensional tori (Liouville theorem) and some *special fibers* that contain all *critical points*, i.e. points where the momentum map (H, K) has $\mathrm{rk} < 2$.

Critical points of (H, K) are not isolated in *regular* $Q^3_h = \{x \mid H(x) = h\}$ ($\mathrm{grad}\, H \neq 0$ in Q^3). They are united in several critical S^1, i.e. closed orbit of $v = \mathrm{sgrad}\, H$. Every critical orbit of v belongs to a special fiber. Recall that a function F on Q^3 satisfies *Morse–Bott condition* (is a *Bott function*) if its restriction f is a Morse function on transversal section to every closed critical orbit S^1 of v in Q^3_h. If first integral K is a Bott function on Q^3 then one can describe trajectories of the system on this level of energy in terms of their 2-dimensional closures and their bifurcations through critical fibers.

Closure of almost every trajectory is a Liouville torus. Fiber-wise neighbourhoods of special fibers were effectively classified by A. Fomenko (classes were called "3-atoms"). They have structure of S^1-fibration. A. Oshemkov classified bases of 3-atoms (called 2-atoms) using so-called f-graphs [2].

The next step was done by A. Fomenko and H. Zieschang [3]. They constructed graph invariant with some labels ("molecule") that classifies Liouville foliations on Q^3. Two manifolds are fiber-wise diffeomorphic iff invariants of systems coincide. Closures of trajectories also have the same structure.

These invariants were calculated for various mechanical and physical systems by many authors. Famous Euler, Lagrange, Kovalevskaya [4] cases of integrability in rigid body dynamics, Klebsh and Steklov integrable cases for a body motion in liquid and new cases of integrability (Sokolov and Bogoyavlenskii cases) are among them. Moreover, some technic (expression of some bases of $H^1(T^2)$ via so-called λ-cycles) helps use this theory for every 3-dimensional fiber-wise submanifold Q^3 of a symplectic manifold M^4 (not only for isoenergy submanifolds Q^3_h).

Some of these cases, i.e. Kovalevskaya and Sokolov, have integrable analogs on orbits of coadjoint representation in the dual space of the Lie algebras $so(3, 1)$ and $so(4)$ [5]. We will present these invariants for Kovalevskaya cases on $so(3, 1)$ and $so(4)$. Topological type (class of diffeomorphisms) of isoenergy submanifolds in the case of $so(4)$ also will be discussed: they were determined without any numerical calculations, only by analysing Fomenko–Zieschang invariants.

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GENERALIZED INTEGRABLE BILLIARDS AND FOMENKO CONJECTURE.

IRINA S. KHARCHEVA

1. Let us consider free motion of a particle in some fixed domain $\Omega \in \mathbb{R}^2$ with elastic reflection on the boundary $P = \partial\Omega$. Thus to square of the velocity vector preserves during motion.

If domain's boundary P is a piece-wise curve and consist of several arcs of confocal ellipses and hyperbolas then such billiard (we call this billiards as elementary billiards) is integrable, i.e. it has an additional first integral Λ. The straight lines containing the segments of the polygonal billiard trajectory are tangents to a certain quadric (ellipse or hyperbola). The parameter of this quadric is the value of the additional integral Λ. Thus the isoenergy surface Q^3_h is foliated by integral Λ and can be described in terms of Fomenko–Zieschang invariants [1].

2. Class of *topological billiards* was constructed by gluing together two elementary domains by their common boundary arc [2]. Produced domain is a covering space upon some flat base. The projection is degenerate only in the points of gluing. This billiard remains integrable and dynamics on such domain is clear. Trajectory changes the sheet of this domain if reaches the glued boundary.

In this case, the billiard is represented as a two-dimensional cell complex. By gluing new cells (elementary billiards) to the boundary, we complicate the topology of the cell complex.

How one can define dynamics if three or more domains are glued together by a common boundary arc? Some permutation σ should be added to this arc: trajectory that starts at the sheet i and reaches this boundary arcs should continue on the "sheet" $\sigma(i)$. Note that the projections of these "sheets" can be both on the same side or on different sides (in $\mathbb{R}^2$) on the projection of this arc.

Such billiards were constructed by V.V. Vedyushkina and called a *billiard books* in [3]. Roughly speaking, we get a "book", where several sheets are glued to the "spine".

3. Analyzing a large number of billiard domains and mechanical systems and comparing their Fomenko–Zieschang invariants A.T. Fomenko formulated the following conjecture:

Let us consider a foliation generated by integrable Hamiltonian system with 2 degrees of freedom on 3-dimensional manifold and classified by Fomenko–Zieschang invariant. Some billiard system with the same Fomenko–Zieschang invariant should exist. It means that some billiard book can be constructed for an arbitrary integrable system s.t. they have the same structure of trajectory closures.

The first part of this conjecture is correct. It means that any typical bifurcation of Liouville tori (fiber-wise neighbourhood of nondegenerate special fiber, called "3-atom") can be realized as special fiber of some billiard book. Effective algorithm of constructing its domain will be presented.

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INTEGRAL TRANSFORMS WITH RESPECT TO THE EULER CHARACTERISTIC INTEGRATION

HUY MAI

Euler Characteristic Integration comes equipped with its own set of integral transforms, which proves to be essential to some recent developments. We will show that the Persistent Homology Transform (and some others) completely characterize compactly supported functions in any Euclidean space. We will also discuss the interactions between the classical Fourier-Sato transform and a certain pseudo-inner product on the space of constructible functions.

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PERSISTENT HOMOLOGY OF KDE FILTRATION ON RIPS COMPLEX

JISU KIM

Abstract. When we observe a point cloud in the Euclidean space, the persistent homology of the upper level sets filtration of the density is one of the most important tools to understand topological features of the data generating distribution. The persistent homology of KDEs (kernel density estimators) for the density function is a natural way to estimate the target quantity. In practice, however, calculating the persistent homology of KDEs on d-dimensional Euclidean spaces requires to approximate the ambient space to a grid, which could be computationally inefficient when the dimension of the ambient space is high or topological features are in different scales. In this paper, we consider the persistent homologies of KDE filtrations on Rips complexes as alternative estimators. We show consistency results for both the persistent homology of the upper level sets filtration of the density and its simplified version. We also describe a novel methodology to construct an asymptotic confidence set based on the bootstrap procedure. Unlike existing procedures, our method does not heavily rely on grid-approximations, scales to higher dimensions, and is adaptive to heterogeneous topological features.

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METRICS FOR PERSISTENCE DIAGRAMS: AN OPTIMAL TRANSPORT VIEW.

THÉO LACOMBE

Persistence diagrams (PDs) appear as a core tool to encode topological information in data analysis. PDs provide a concise way to summarize the underlying topology of a given object at all scales—informally as a locally finite point cloud supported on the upper half-plane $\{(t_1, t_2) \in \mathbb{R}^2, t_2 > t_1\}$. The space of these diagrams can be equipped with partial matching metrics, with theoretical guarantees on the stability of diagrams under pertubations of input data. However, the computational cost of such metrics is known to be prohibitive in large-scale applications, and the structure these metrics induce on the space of PDs is non-linear. This makes the use of standard statistical tools or machine learning techniques—even as simple as estimating Fréchet means or barycenters of a sample of PDs—challenging. We present a way to address these issues by reformulating PD metrics as an optimal *partial* transport problem [1], and show how recent advances in computational optimal transport [2, 3, 4] can be adapted to deal efficiently with large samples of PDs. In particular, regularizing the optimal transport problem with an entropic penalization yields a convex problem that can be solved efficiently with the Sinkhorn algorithm. Unlike previous methods to approximate PD metrics, this algorithm can be parallelized and implemented efficiently on GPUs. These approximations are also differentiable, leading to a simple and scalable method to estimate barycenters of PD samples. We showcase the strength of this approach by estimating the Fréchet means and performing k-mean clustering with diagram metrics on large PD samples [5].

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RIPS MAGNITUDE

DEJAN GOVC

Magnitude [1] is a numerical invariant of metric spaces (and more generally, enriched categories [4]) introduced by Tom Leinster which has been shown to arise as the graded Euler characteristic of a certain homology theory [3]. Richard Hepworth has recently suggested to examine an analogous invariant for persistent homology, called Rips magnitude, which arises as a graded Euler characteristic of persistent homology. In the talk I will describe some of its basic properties and examine its asymptotic behaviour in the case of finite subsets of the circle, using a result of Adamaszek [2].

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MULTIPARAMETER PERSISTENCE LANDSCAPES

OLIVER VIPOND

An important problem in the field of Topological Data Analysis is defining topological summaries which can be combined with traditional data analytic tools. In recent work Bubenik introduced the persistence landscape, a stable representation of persistence diagrams amenable to statistical analysis and machine learning tools. In this talk we generalise the persistence landscape to multiparameter persistence modules providing a stable representation of the rank invariant. We show that multiparameter landscapes are stable with respect to the interleaving distance and persistence weighted Wasserstein distance, and that the collection of multiparameter landscapes faithfully represents the rank invariant. Finally we provide example calculations and statistical tests to demonstrate a range of potential applications and how one can interpret the landscapes associated to a multiparameter module.

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EARLY WARNING SIGNAL FOR FLOODS USING PERSISTENT HOMOLOGY

SYED MOHD SADIQ SYED MUSA

Flooding is an environmental hazard that occurs almost everywhere around the world and it contributes to a high number of deaths and loss of properties. Analysis of streamflow data can give us important climatic information for flooding events. Persistent homology (PH), a tool in topological data analysis (TDA) provides a new way to look at the information in a data set using a qualitative approach. PH uses topology to extract qualitative information from noisy data sets at various scale of the data by giving information on topological features that exist in the data set. In this paper, we present a new approach for streamflow data analysis by using PH. An analysis was conducted at the Guillemard Bridge Station, Kelantan River, Malaysia. The topological features extracted are summarize in a topological summary known as the persistence landscape. By analysing the persistence landscape, we get a signal that can be use for an early warning signal for floods. The result shows that this signal exhibit critical slowing down when approaching flood events. Increase in variance and power spectrum are the indicators for this critical slowing down. As a conclusion, this study suggests that the information on topological features of streamflow data can be used as a basis for an early warning signal for floods.

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THOUGHTS ON SECTIONAL CATEGORY AND RELATIVE COHOMOLOGY

ARTURO ESPINOSA BARO

Joint work with Z. Błaszczyk and J. Carrasquel (Adam Mickiewicz, Poland).

The Lusternik-Schnirelmann category of a space, and Farber's topological complexity [1], are particular examples of a more general notion, the sectional category, introduced by Schwarz in [2]. A famous theorem due to Eilenberg and Ganea, [3] gives a characterization of the LS category of an aspherical space as the cohomological dimension of its fundamental group (for dimension greater than 3). In the context of topological complexity, the generalization of the theorem of Eilenberg and Ganea, or any other algebraic characterization of the TC of aspherical spaces, remain as one of the most interesting open problems. Recently, in [4] Farber, Grant, Lupton and Oprea have used the tools of Bredon equivariant cohomology, developed by Bredon in [5], to offer new cohomological bounds for topological complexity of aspherical spaces. We consider a different cohomology theory, a notion of relative cohomology due to Adamson, [6]. This notion has the advantage of being bounded above by Bredon, so it is interesting to consider it as a candidate for a finer bound for TC of groups. We will study the relationship with the Bredon one, and whether if it is possible to obtain relevant information for sectional category of subgroups inclusions.

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CLUSTER ANALYSIS OF HAZE EPISODES BASED ON TOPOLOGICAL FEATURES

NUR FARIHA SYAQINA ZULKEPLI

Persistent homology is a tool used in topological data analysis (TDA) to extract essential topological features from data. Cluster analysis is a technique that is used for grouping objects in data sets into different clusters such that the members that are placed in the same cluster are similar with each other compared to the members in other clusters. Commonly, cluster analysis is applied based on available information of data without considering topological information. Thus, this study aims to apply cluster analysis based on topological features and the effectiveness of this approach is observed by comparing with original clustering approach. This is achieved by extracting topological features (connected components and holes) of particulate matter (PM_{10}) which is the major pollutant during haze episodes in Malaysia and the cluster members (months with and without haze) are observed. We apply Hierarchical Agglomerative Clustering Analysis (HACA), which is a standard technique in air quality studies, on its own (baseline) and the results are compared with combination of HACA and topological features (proposed). HACA process is initiated by calculating dissimilarity distance between objects (months) and two objects with minimum distance is merged forming a single cluster. For the next cluster, new set of distance is calculated and again clusters with minimum distance merged and form a cluster. This process is continued in hierarchical way until one single cluster containing all objects is produced. Based on the results, proposed approach is able to cluster months with and without haze correctly compared with baseline approach.

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PERSISTENT HOMOLOGY ON MALAYSIAN DATA SETS

FATIMAH ABDUL RAZAK

In this new age of data empowered race, many different methods claim to uncover hidden structures and information from data sets. Algebraic topology assigns algebraic invariants such as groups and vector spaces to topological space. In particular, homology theory is used to detect topological features such as components, holes and voids. Persistent homology is a method used on data sets to detect these topological features.

Persistent homology is applied on time series by first utilizing Taken's theorem in order to get higher dimensional sets of data. The aim is to develop an early warning systems for floods and haze (both occuring annually in Malaysia) as well as financial crashes by detecting extreme changes in the topological shapes of datasets. To this end, we apply persistent homology to time series of river streamflows, atmospheric content such as Particulate Matter less than 10 micrometers (PM_{10}) as well as some Asian Financial stock market indicators across a few decades.

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SIMPLICIAL KIRCHHOFF INDEX

WOONG KOOK AND KANG-JU LEE

We introduce a high-dimensional analogue of Kirchhoff index which is also known as total effective resistance. This analogue, which we call simplicial Kirchhoff index $Kf(X)$, is defined to be the sum of simplicial effective resistances of all $(d+1)$-subsets of the vertex set of a simplicial complex X of dimension d. For a d-dimensional simplicial complex X with n vertices, we give formulas for simplicial Kirchhoff index in terms of the pseudo inverse of the Laplacian L in dimension d and its eigenvalues:

$$Kf(X) = n \cdot \operatorname{tr} L^+ = n \cdot \sum_{\lambda \in \Lambda_+} \frac{1}{\lambda}$$

where L^+ is the pseudo-inverse of L, and Λ_+ is the set of non-zero eigenvalues of L. Using this formula, we obtain an inequality for a high-dimensional analogue of algebraic connectivity and Kirchhoff index, and propose these quantities as measures of robustness of simplicial complexes. In addition, we derive its integral formula and relate this index to a simplicial dynamical system.

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LARGE RANDOM SIMPLICIAL COMPLEXES

LEWIS MEAD

Random simplicial complexes extend the highly studied Erdős-Rényi model for random graphs to a high-dimensional analogue and have been increasingly studied over the past 15 years. In this talk I will introduce general models of random simplicial complexes which are constructed from a random hypergraph process. The general models presented in this talk include other well studied probabilistic models of random simplicial complexes from Costa-Farber [1], Kahle [2], and Linial-Meshulam [3] as special cases. Added generality in these new models introduces further complications and difficulties to fully understand the structure beneath. However elementary steps to pin down random topological properties such as estimating face numbers, connectivity thresholds, describing Betti numbers, and a duality between the models has been achieved. The talk will conclude with plans of some future work and interesting open questions. This talk is based on joint work with Michael Farber and Tahl Nowik.

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PERSISTENCE CURVES: A NEW VECTORIZATION OF PERSISTENCE DIAGRAMS

YU-MIN CHUNG[1] AND AUSTIN LAWSON[1]

Persistence diagrams are a main tool in the field of Topological Data Analysis (TDA). They contain fruitful information about shapes of underlying objects. However, performing machine learning algorithms or statistical methods directly on persistence diagrams is a challenging problem due to the limitation of the space of persistence diagrams. For that reason, summarizing and vectorizing these diagrams is an important topic currently researched in TDA ([2, 1]). In this work, we develop a new way of summarizing diagrams: *Persistence Curves* (PC), and show practical uses of PC to several texture datasets.

The first part of the work devote to the foundation and theory of PCs. The main construction of PCs comes from the Fundamental Lemma of Persistent Homology, which reveals Betti numbers from persistence diagrams. As an example, Euler Characteristics Curve (ECC) is a special case of PC. PCs are family of curves and hence they can be used in a variety of situations depending on the data. We prove a rigorous bound for a general family of PCs. In particular, certain family of PCs admit the stability property. Furthermore, we show that Persistence Landscapes (PL) are special cases of PCs. PC provides the bridge from the classical ECC to modern PL.

The second part of the work is to apply PCs to real world applications. We investigate classifications of texture images on the three well-know texture datasets: Outex [4], UIUCTex [5], and KHT-TIP [7], where sample images are shown in Figure 1. Our results outperform some of TDA methods [3, 6] that applied to Outex. The performances for UIUCtex and KTH also reveal strong evidence. PCs are intrinsic characteristics of textures. Finally, we will show that PCs are simple and intuitive to implement.

(A) Outex (B) UIUCtex (C) KTH-Tips

FIGURE 1. Snapshots of the texture databases. Our best classification rate for each database are 99%, 92.4%, and 91.5%, respectively.

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COMPARISON THEOREMS OF PHYLOGENETIC SPACES AND ALGEBRAIC FANS

YINGYING WU

With recent developments in the acquisition of biological data and progress in genetics, biology has become a data-rich discipline; for example, biologists have wielded CRISPR to track a mammal's development from a single egg into an embryo with millions of cells [5], which creates a demand for a deeper understanding of evolutionary histories. I will report my results on comparison theorems between phylogenetic spaces that represent evolutionary histories and algebraic fans over simplicial complexes which arise in the moduli space of smooth marked del Pezzo surfaces. I will show homeomorphisms between their projective spaces and simplicial complexes formed by root subsystems. Furthermore, I will present embeddings between spaces of phylogenetic trees and networks, and that between the projective spaces of phylogenetic trees and networks. Knowing the correspondence between mathematics and genomic structures may expedite the discovery of the missing pieces in biology whose counterparts are naturally expected in mathematics, and equip investigations in phylogeny with more mathematical tools from algebraic geometry and tropical geometry [7, 3, 6].

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PERSISTENT HOMOLOGY OF RANDOM ČECH COMPLEXES ON MANIFOLDS

AKSHAY GOEL

The emerging research area known as random topology, motivated by many issues in manifold learning and Topological Data Analysis (TDA), comprises theoretical results that characterize the asymptotic behavior of topological properties of random objects. One aspect of this area is the study of random geometric complexes and their topological properties called Betti numbers and persistent Betti numbers. In this work, we concentrated on a typical type of random geometric complex, known as random Čech complex, denoted by $C(\mathcal{X}_n, r_n)$, if constructed on a finite set of points $\mathcal{X}_n = \{x_1, x_2, \dots, x_n\} \in \mathbb{R}^d$ and the radius is $r_n > 0$. Here, $\{r_n\}$ is a non-random sequence of positive numbers tending to zero for which three regimes (sparse regime, thermodynamic regime and dense regime) are divided according to the limit of $\{n^{1/m} r_n\}$: zero, finite or infinite, where $m \le d$ is the intrinsic dimension of the space. It is known that the limiting behavior of Betti numbers in each regime is totally different.

We establish the strong law of large numbers for Betti numbers of random Čech complexes built on $\mathbb{R}^N$-valued binomial point processes in the thermodynamic regime [1]. Here we consider the case where the underlying distribution of the point processes is supported on a C^1 m-dimensional compact manifold embedded in $\mathbb{R}^d$. This result is new since only lower and upper bounds for the expectation of Betti numbers were known in the thermodynamic regime[3]. Moreover, from the applications point of view especially in TDA, considering only homology is not enough. It is important to see how persistent the 'holes' are, which constitutes the theory of persistent homology. We also extend our result for Betti numbers to persistent Betti numbers, and hence to persistence diagrams due to [2]. Here persistence diagram is regarded as a counting measure rather than as a muliset. All these results are proved under very mild assumption which only requires that the common probability density function belongs to L^p spaces, for all $1 \le p < \infty$.

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AUSLANDER-REITEN GRAPH DISTANCE AS A BOTTLENECK METRIC

KILLIAN MEEHAN

In collaboration with David Meyer.

This project investigates the potential of quiver theoretic bottleneck metrics for use over non-totally-ordered posets. The classical bottleneck metric on persistence diagrams is discussed as a diagonal interleaving metric, as are various modifications to these familiar notions for the $\mathbb{A}_n$ quiver (zig-zag) setting. Following this, stability results for the quiver theoretic bottleneck metrics are presented relative to their classical counterparts.

Central to our construction is the use of the Auslander-Reiten (AR) quiver for arbitrary orientations of $\mathbb{A}_n$. I present a formulaic representation of the AR quiver in this setting, derived from the Knitting Algorithm, but with the advantage that it conveys the full structure without the sequential construction required by the Knitting Algorithm.

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HARMONIC CYCLES AND RATIONAL WINDING NUMBERS

YOUNNG-JIN KIM AND WOONG KOOK

In this presentation, we discuss high-dimensional harmonic cycles. A harmonic cycle λ is a discrete harmonic form, i.e., a solution of the Laplacian equation

$$\triangle_n \lambda = 0 \tag{0.1}$$

with the Laplacian operator

$$\triangle_n = \partial_{n+1}\partial_{n+1}^t + \partial_n^t \partial_n \tag{0.2}$$

obtained from the chain complex $\partial_i : C_i(X) \to C_{i-1}(X)$ of a cell complex X. By the combinatorial Hodge theory, harmonic spaces are isomorphic to the homology groups with real coefficients. In particular, an acyclic cell complex has only the trivial harmonic cycle. In our talk, we will mainly address the case

$$\operatorname{rk} \widetilde{H}_n(X) = 1,\ \operatorname{rk} \widetilde{H}_{n-1}(X) = 0 \text{ and } \operatorname{rk} \widetilde{H}_{n+1}(X) = 0, \tag{0.3}$$

and introduce a formula for the *standard harmonic cycle* λ as a generator of the harmonic space,

$$\lambda = \sum_Y w(C_Y) C_Y \tag{0.4}$$

where the summation is over the cycletrees Y with its minimal cycle C_Y, and $w(\cdot)$ is the winding number map. We will also discuss intriguing combinatorial properties of λ with respect to (dual) spanning trees, (dual) cycletrees, winding number $w(\cdot)$ and cutting number $c(\cdot)$, i.e., for example,

$$\lambda \circ z = k_n(X) w(z) \text{ and } \lambda \circ z = k^n(X) c(z) \tag{0.5}$$

where $k_n(X)$ is the n-th tree number and $k^n(X)$ is the n-th dual tree number.

Furthermore, we will present an application for detecting the oscillation in flows for a periodic dynamical system with random perturbations through a simple example.

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PERCOLATION ON HOMOLOGY GENERATORS IN CODIMENSION ONE

TATSUYA MIKAMI

Percolation theory is a branch of probability theory which describes the behavior of clusters in a random graph, and it has many applications to material science such as immersion in a porous stone. Recently, craze formation in polymer materials is gaining attention as a new type of percolation phenomenon. The paper [2] shows that a large void corresponding to a craze of the polymer starts to appear by the process of coalescence of many small voids, which suggests that "percolation of nanovoids" is the key mechanism to initiate craze formation.

In this talk, I introduce a new percolation model motivated from the craze formation of polymer materials. For the sake of modeling the coalescence of nanovoids, this model focuses on clusters of holes in $\mathbb{R}^d$ as higher dimensional topological objects, while the classical percolation theory mainly studies clusters of vertices (i.e., 0-dimensional objects). More precisely, this model uses homology generators in dimension $d - 1$ for representing the holes, and the behavior of clusters of those holes are studied. This talk is based on the paper [1].

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SECTIONAL CATEGORY À LA QUILLEN

JOSÉ GABRIEL CARRASQUEL VERA

Joint work with U. Buijs (Málaga, Spain) and L. Vandembroucq (Minho, Portugal).

The Lusternik-Schnirelmann category of a space is a particular case of a more general invariant of maps, introduced by Schwarz [7], called the *sectional category*:

$$cat(X) = \mathrm{secat}(* \hookrightarrow X).$$

Farber's Topological complexity [3] is also a particular case of sectional category, namely, it is the sectional category of the diagonal inclusion:

$$\mathrm{TC}(X) = \mathrm{secat}(X \hookrightarrow X \times X).$$

A *rational* space is a topological space whose homotopy groups are vector spaces over the rational numbers. To any nilpotent space X we can assign its *rationalisation map*, $\rho : X \to X_0$, where X_0 is a rational space and $\pi_*(\rho) \otimes \mathbb{Q}$ (or equivalently $H(\rho, \mathbb{Q})$) is an isomorphism. We can think of X_0 as a space capturing the *torsion free* information of X.

In both Sullivan's [8] and Quillen's [6] approach a functor $F\colon \mathbf{Top} \to \mathcal{A}$ is constructed, being $\mathcal{A}$ the category of commutative differential graded algebras or differential graded Lie algebras, respectively. These functors restricted to the category of finite type rational spaces turn out to be equivalences of homotopy categories. This means that the rational (torsion free) homotopy type of X is completely encoded algebraically in $F(X)$!

This is very useful because it permits us to study any rational homotopy invariant in purely algebraic terms. We therefore speak of $F(X)$ (and any object equivalent to it) as a *model* for X. In particular, algebraic methods for computing invariants of the type of topological complexity can be developed. An example of this is the main result of [2] where we give a purely algebraic characterisation of sectional category.

The study of sectional category for rational spaces has been done using only Sullivan minimal models. The reason for this is that they are ideal objects for modelling products and fibrations: the tensor product of minimal models is the model of the cartesian product. For Quillen models the situation is much more difficult. In [9], D. Tanré gave a way of constructing the minimal Quillen model for the product of spaces, where the construction of the differential is not explicit. Later on, G. Lupton and S. Smith gave an explicit differential for this model in the case that one of the factors is a co-h-space[5].

In our talk, we will develop techniques to study sectional category using Quillen models. For this it is crucial to find explicit Quillen models for products and diagonal inclusions.

This is done through the *infinity Quillen functor* introduced in [1]. This functor assigns to a commutative differential graded algebra (cdga) model of a space the minimal Quillen model of the space. The construction consists on dualizing the cdga model for the space to make a co-commutative differential graded coalgebra, then transfer this structure through a retract to get an C-infinity coalgebra structure on the rational homology of the space, which translates into an explicit differential of the

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minimal Quillen model.

Then we give a characterisation of LS category and sectional category through Quillen models using the Whitehead characterisation and a model for the fat-wedge.

We will outline possible applications to some open problems for rational sectional category. For instance, the relation between the sectional category of a map and the LS category of its homotopy cofibre [4] or the Ganea conjecture for rational topological complexity.

Lastly we will expose the computational tools that we have implemented in order to carry out tedious computations.

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TOPOLOGICAL COMPLEXITY AND EFFICIENCY OF MOTION PLANNING ALGORITHMS

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A motion planner in a space X is an algorithm which, given a pair of points $(x, y) \in X \times X$, outputs a path in X with initial point x and terminal point y. This notion is usually considered in the context of robotics, where X is taken to be the space of all states ("configuration space") of a mechanical system. One would hope for a motion planner that is stable in the sense that a minor change of either the initial or terminal state results in a predictable change of the path taken by the mechanical system. This, however, turns out to be rarely possible. In order to quantify the "order of instability" of configuration spaces of mechanical systems, Farber [2] introduced the notion of topological complexity.

A shortcoming of Farber's approach is that it does not take into consideration any notion of efficiency of motion planners, e.g. measured in terms of covered distance or spent energy. Yet motion planners which do not comply with basic constraints (e.g. a path from any state to itself is constant) should be ruled out as inadequate. Also, one would like to be able to quantify efficiency of motion planners in order to compare them.

I will discuss a variant of Farber's topological complexity, defined for smooth compact Riemannian manifolds, which addresses the problem hinted at above by taking into account only motion planners with the lowest possible "average length" of the output paths.

The talk is based on joint work with J. G. Carrasquel Vera.

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STREAMING ALGORITHM FOR EULER CHARACTERISTIC CURVES OF MULTIDIMENSIONAL IMAGES

TERESA HEISS

In various applications, including material science, medical imaging, and astrophysics, there is need to analyze high resolution images coming from various types of scanners. In particular, modern micro-CT-scanners produce three-dimensional images with up to 10^{12} voxels. Available implementations of topological descriptors, including persistent homology, are not efficient enough to handle such datasets. As an alternative, we propose a simpler topological descriptor, namely the Euler characteristic curve.

Viewing a gray scale image as a function from the voxels to the gray intensity values, the Euler characteristic curve of the image maps each such value to the Euler characteristic of the corresponding sublevel set. The Euler characteristic curve can be seen as a summary of the Betti curves as well as a summary of persistent homology.

We developed the first algorithm to compute the Euler characteristic curve of images of arbitrary dimension that is time- and memory-efficient enough to handle images with more than 10^{12} voxels [1]. The software—CHUNKYEuler—is available as open source: https://bitbucket.org/hubwag/chunkyeuler.

Joint work with Hubert Wagner.

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