POINT VORTEX DYNAMICS ON MINIMAL SURFACES

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In fluid dynamics on curved surfaces, the shape of the flow field is a primary factor in determining the dynamics of the fluid flow. Some mathematical models of fluid equations on surfaces are proposed in terms of differential geometry without physical experiments [1]. Our purpose is to characterize dynamical properties derived from each of these models in terms of geometric properties of the surfaces towards validating the model with a physical experiment.

Let us consider a situation where fluid dynamics on curved surfaces can be physically realized. As we know, when we hang some wires and dip them into soap solution, a soap film spanning the wires is formed [2]. The shape of the soap film is determined by the Young-Laplace equation. Assuming the pressure difference across a film is equal to zero, we can deduce that the shape of the soap film is given as a minimal surface. The motion of fluid in the soap film is governed by the Euler equations on surfaces when we assume the fluid is incompressible and inviscid. In this talk, we treat the motion of fluid in a soap film as that of incompressible and inviscid fluid on a minimal surface. When we compare physical experiments with theoretical analysis of the Euler equations on a minimal surface, it is necessary to construct a numerical scheme. To this end, we divide this task into two parts: surface registration of a minimal surface and numerical computation of the Euler equations, namely, the boundary value problem to determine a minimal surface and its conformally flat domain with a prescribed boundary configuration and an initial boundary value problem for the Euler equations on the minimal surface with the no-normal boundary condition.

In surface registration, it is crucial for computing differential equations on surfaces to choose a ”good” parametrization and a parameter space. When we choose a uniformizing chart as the parametrization, we can represent every metric by $\lambda^2((dx^1)^2 + (dx^2)^2)$ for some positive function λ on the conformally flat domain which enables us to use the simplified coordinate representation of the differential equations as well as that of a geometric quantity such as the Gaussian curvature. Since the uniformizing chart is defined on the whole space of the surface, we can compute the differential equations on the domain without swapping charts as often as the domain which we focus on changes. From these reasons, let us numerically construct a uniformizing chart on a given surface. The normalized Ricci flow on a surface is helpful for our purpose. The normalized Ricci flow is defined as an evolution equation of Riemannian metrics and preserves the conformal structure and the area of an initial metric. Moreover, the solution of the normalized Ricci flow exponentially converges to a constant curvature metric in smooth topology. Hence we can obtain the conformally flat domain from the long-time limit of the solution. Ricci flow on not only a surface but also discrete one has been recently investigated and applied to numerical computation of a uniformizing chart. For general reference see [3]. In this talk, we provide a numerical scheme with high accuracy by using discrete Ricci flow and focusing on a geometric property of a minimal surface. Another key device in the scheme is the method of fundamental solution (MFS), which is a meshfree numerical solver for linear elliptic partial differential equations [4].

In numerical computation of the Euler equations, we adopt the vortex method as the numerical solver [5]. Thanks to the invariance of the vorticity along a fluid particles, discretizing an initial vorticity distribution with a liner combination of delta functions, called point vortices, we can treat the velocity field as a finite-dimensional Hamiltonian vector field whose Hamiltonian consists of a Green’s function for Laplacian on the surface and the regularized Green’s function by the geodesic distance, called Robin function. In particular, we examine dynamical evolution of point vortices on a minimal surface. In order to carry out theoretical analysis, assuming the existence of a non-trivial Killing vector field on a minimal surface, we provide exact solutions of the Hamiltonian system as an

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application of [6]. After introducing we introduce a numerical scheme for the Green's function and the Robin function by using MFS, we compare numerical results with the exact solutions. Finally, we investigate dynamical behavior of point vortices in terms of a boundary configuration and shape of a minimal surface by using the proposed numerical solver. This talk is based on a joint work with Dr. Koya Sakakibara (Kyoto University).

References

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