A COMPUTATIONAL FRAMEWORK FOR CONNECTION MATRIX THEORY

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Abstract. Algebraic topology and dynamical systems are intimately related: the algebra may constrain or force the existence of certain dynamics. Morse homology is the prototypical theory grounded in this observation. Conley theory is a far-reaching topological generalization of Morse theory and the last few decades have seen the development of a computational version of the Conley theory. The computational Conley theory is a blend of combinatorics, order theory and algebraic topology and has proven effective in tackling problems within dynamical systems.

Within the Conley theory the connection matrix is the mathematical object which transforms the approach into a truly homological theory; it is the Conley-theoretic generalization of the Morse boundary operator. We'll discuss a new formulation of the connection matrix theory, which casts the connection matrix in categorical, homotopy-theoretic language. This enables the efficient computation of connection matrices via the technique of reductions in combination with algebraic-discrete Morse theory. We will also discuss a software package for such computations. Time permitting, we'll demonstrate our techniques with an application of the theory and software to the setting of transversality models [9]. This application allows us to compute connection matrices for the classical examples of Franzosa [5] and Reineck [13] as well as high-dimensional examples from a Morse theory on spaces of braid diagrams introduced in [6].

Introduction

Topology and algebraic invariants have played a prolific role in dynamical systems [1, 16]. Loosely stated, a dynamical system engenders topological data: both local (e.g. fixed points) and global (e.g. attractors). The topological data have associated algebraic invariants (e.g. homology) and the relationship between local and global is codified in the algebra.

Morse theory is an influential instantiation of this idea wherein the local data (nondegenerate fixed points) in the gradient flow $\dot{x}(t) = -\nabla f(x(t))$ of generic map $f : M \to \mathbb{R}$ are graded by their Morse index and contribute to a chain complex $(C_\bullet, \partial)$. The boundary operator is determined by the structure of the connecting orbits. It is classical that the Morse homology $H_\bullet(C_\bullet, \partial)$ is isomorphic to the singular homology $H_\bullet(M)$. Conley theory is a purely topological generalization of Morse theory: the index of an isolated invariant set is a topological space whose homology gives a coarse description of the unstable dynamics. Essential to the Conley index is the property of *continuation*: the index is robust to perturbations of the system [1]. Our recent work [7] concerns developing a categorical, homotopy-theoretic framework for the computation of connection matrices, the Conley-theoretic generalization of the Morse boundary operator [5]. We outline a computational connection matrix theory and give application to transversality models in [9]. Moreover in [8] we give the specifics on the particulars of the algorithm, including a novel scheme for an implicit discrete Morse theory on cubical complexes.

Computation of Connection Matrices

Analogous to the Morse boundary operator, the *connection matrix* is a boundary operator defined on Conley indices [5]. In contrast to the Morse boundary operator, the connection matrix is not obtained directly from the trajectories, but it is related to them. This relationship implies the basic utility of a connection matrix is to prove existence of connecting orbits [10]. At a higher level, it serves as an algebraic representation of global dynamics and may used in some cases to construct semi-conjugacies of the global attractor [3, 12]. Ultimately, the connection matrix completes the Conley theory to a homological theory [11] for dynamical systems.

In recent work [7] we gave a categorical, homotopy-theoretic treatment of the connection matrix theory. In this setting we can interpret a connection matrix as the boundary operator of a particularly simple representative of an isomorphism class in an appropriate homotopy category. We show that, in the case of fields, the use of homotopy categories enables the connection matrix theory to be made functorial.

Using the homotopy-theoretic framework, the computation of a connection matrix can be phrased in terms of (filtered) reductions, a technique introduced in [4] and extensively used in [15]. In the case of the computational Conley theory, where the typical input is a decomposition of a cell complex into attracting blocks, we show that discrete Morse theory induces a reduction and can be used to provide an efficient algorithm for computing connection matrices. This provides a purely algorithmic and constructive proof of existence of connection matrices [5, 14]. Moreover, we'll discuss publicly available software packages for the connection matrix theory [2].

Application to a Morse Theory on Braids

As developed in [6], a set $\{u^n(t,x)\}$ of solutions to a scalar parabolic partial differential equation of the form $u_t = u_{xx} + f(u_x, u, x)$ may be lifted to (x, u, u_x)-space to create a braid. The space of braids partitions into isotopy braid classes and monotonicity properties of the PDE induce dynamics on braid classes. Discretized braids (whose strands are piecewise linear) are a finite-dimensional approximation to the space of braids. In this case the phase space partitions into a cubical complex of discrete braid classes. The parabolic PDE induces dynamics on braid classes via the comparison principle, which leads to the notion of a Conley index for braid classes. We will discuss applications of the algorithms to compute connection matrices in this setting [9], including examples of 10 and 12-dimensional cubical complexes. The insights obtained from these high-dimensional computations have led to new conjectures for the theory [9].

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