# RANK INVARIANT FOR ZIGZAG MODULES

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The rank invariant [2] is of great interest in studying standard persistence modules over $\mathbb{R}^n$ [3, 4, 5, 6, 7]. In particular, it is well known that

(a) the rank invariant is a complete invariant for (standard) persistence modules over $\mathbb{R}$, and
(b) for $n \geq 2$ the rank invariant fails to be complete for (standard) persistence modules over $\mathbb{R}^n$.

Motivated by these facts, in our work, we wondered whether it would be possible to define a suitable notion of rank invariant for *zigzag* modules, and whether the resulting invariant would be complete.

We elucidate such a generalization of the rank invariant to zigzag modules by establishing a certain relationship between the limits and colimits of every subdiagram of a zigzag module. Following the formulation of [1], we consider zigzag modules represented over $\mathbb{R}$. Then, given a zigzag module $M$ over $\mathbb{R}$, for every $s \leq t$ in $\mathbb{R}$, we study the canonically induced map from the limit to the colimit of the subdiagram $M|_{[s,t]}$:

$$\phi_M(s,t) : \varprojlim M|_{[s,t]} \longrightarrow \varinjlim M|_{[s,t]}.$$

Then, the rank invariant function $\mathrm{rk}(M) : \{(s,t) \in \mathbb{R}^2 : s \leq t\} \to \mathbb{Z}$ associated to $M$ is defined by

$$\mathrm{rk}(M)(s,t) := \mathrm{rank}\left(\phi_M(s,t)\right).$$

In particular, we prove that the rank invariant function of a zigzag module recovers its interval decomposition. Our results therefore imply that *our rank invariant is a complete invariant for zigzag modules.*

Complementing the characterization result mentioned above, we also show that the erosion distance (as in the work of A. Patel [6]) between the rank invariant functions associated to two arbitrary zigzag modules is bounded from above by the interleaving distance [1] between the zigzag modules (up to a multiplicative constant).

As a further extension, our construction allows us to extend the notion of generalized persistence diagram by A. Patel [6] to zigzag modules valued in any symmetric monoidal bicomplete category.

**NOTE:** A preprint with a full description of these ideas is available in our recent arxiv preprint `https://arxiv.org/abs/1810.11517`.

## References

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