GENERALIZED INTEGRABLE BILLIARDS AND FOMENKO CONJECTURE.

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1. Let us consider free motion of a particle in some fixed domain $\Omega \in \mathbb{R}^2$ with elastic reflection on the boundary $P = \partial\Omega$. Thus to square of the velocity vector preserves during motion.

If domain's boundary P is a piece-wise curve and consist of several arcs of confocal ellipses and hyperbolas then such billiard (we call this billiards as elementary billiards) is integrable, i.e. it has an additional first integral Λ. The straight lines containing the segments of the polygonal billiard trajectory are tangents to a certain quadric (ellipse or hyperbola). The parameter of this quadric is the value of the additional integral Λ. Thus the isoenergy surface Q^3_h is foliated by integral Λ and can be described in terms of Fomenko–Zieschang invariants [1].

2. Class of *topological billiards* was constructed by gluing together two elementary domains by their common boundary arc [2]. Produced domain is a covering space upon some flat base. The projection is degenerate only in the points of gluing. This billiard remains integrable and dynamics on such domain is clear. Trajectory changes the sheet of this domain if reaches the glued boundary.

In this case, the billiard is represented as a two-dimensional cell complex. By gluing new cells (elementary billiards) to the boundary, we complicate the topology of the cell complex.

How one can define dynamics if three or more domains are glued together by a common boundary arc? Some permutation σ should be added to this arc: trajectory that starts at the sheet i and reaches this boundary arcs should continue on the "sheet" $\sigma(i)$. Note that the projections of these "sheets" can be both on the same side or on different sides (in $\mathbb{R}^2$) on the projection of this arc.

Such billiards were constructed by V.V. Vedyushkina and called a *billiard books* in [3]. Roughly speaking, we get a "book", where several sheets are glued to the "spine".

3. Analyzing a large number of billiard domains and mechanical systems and comparing their Fomenko–Zieschang invariants A.T. Fomenko formulated the following conjecture:

Let us consider a foliation generated by integrable Hamiltonian system with 2 degrees of freedom on 3-dimensional manifold and classified by Fomenko–Zieschang invariant. Some billiard system with the same Fomenko–Zieschang invariant should exist. It means that some billiard book can be constructed for an arbitrary integrable system s.t. they have the same structure of trajectory closures.

The first part of this conjecture is correct. It means that any typical bifurcation of Liouville tori (fiber-wise neighbourhood of nondegenerate special fiber, called "3-atom") can be realized as special fiber of some billiard book. Effective algorithm of constructing its domain will be presented.

References

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